12

Mixed finite element methods for second order elliptic problem

12.1 A simple example: Darcy's law

For the model problem,

$$\begin{cases} -\mathrm{div}(\alpha \mathrm{grad}\ u) = f, & \in \Omega, \\ u = 0, & \in \partial\Omega. \end{cases}$$

We can introduce an intermediate variable:

$$p = \alpha \mathrm{grad}\ u.$$

The problem becomes

$$\begin{cases} \alpha^{-1} p - \mathrm{grad}\ u = 0 \\ \mathrm{div} p = -f \end{cases} \tag{12.1}$$

Let

$$Q = H(\mathrm{div}, \Omega), \quad V = L^2(\Omega)$$

Then we have the corresponding variational form of mixed type:

$$\begin{cases} a(p, q) + b(q, u) = 0 & \forall q \in Q, \\ b(p, v) = -(f, v) & \forall v \in V, \end{cases} \tag{12.2}$$

where

$$a(p, q) = (\alpha^{-1} p, q), \quad b(p, v) = (\mathrm{div} p, v).$$

The mixed formulation requires $p \in H(div)$ and $u \in L^2$.

We use the discrete spaces H_h^{div} and L_h^2 to approximate Q and V. Let

$$Q_h = H_h^{\mathrm{div}}(\Omega) \cap H_0(\mathrm{div}, \Omega), \quad V_h = L_h^2(\Omega)$$

Find $p_h \in Q_h, u_h \in V_h$ such that

$$\begin{cases} (\alpha^{-1} p_h, q_h) + (\mathrm{div} q_h, u_h) = 0 & \forall q_h \in Q_h, \\ (\mathrm{div} p_h, v_h) = -(f, v_h) & \forall v_h \in V_h, \end{cases} \tag{12.3}$$

Let the basis function in Q_h be $\{\phi_i\}_{i=1}^n$ and the basis function in V_h be $\{\psi_i\}_{i=1}^m$. Then

$$p_h = \sum_{i=1}^{n} p_i \phi_i \text{ and } u_h = \sum_{i=1}^{m} u_i \psi_i.$$

(12.4)
$$\begin{cases} \sum_{j=1}^{n}(\alpha^{-1}\phi_i,\phi_j)p_j + \sum_{k=1}^{m}(\mathrm{div}\phi_i,\psi_k)u_k = 0, \\ \sum_{j=1}^{n}(\psi_i,\mathrm{div}\phi_j)p_j = -(f,\psi_i), \end{cases}$$

Denote $\bar{p} = (p_1, ..., p_n)$, $\bar{u} = (u_1, ..., u_m)$ and $\bar{x} = (p, u)^T$. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$ such that

$$A_{ij} = (\alpha^{-1}\phi_i, \phi_j),$$

$$B_{ij} = (\psi_i, \mathrm{div}\phi_j),$$

$$\bar{A} = \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}.$$

$b_i = -(f, \psi_i)$, $b = (b_1, ..., b_m)$ and $\bar{b} = [0_{1\times n} f]^T$. For this discrete problem, it can be represented as

$$\bar{A}\bar{x} = \bar{b}.$$

For this discrete problem, we can prove that there exists a unique the solution.

Lemma 86. *The exists an unique* $(p_h, u_h) \in Q_h \times V_h$ *satisfies* (12.68).

Proof. The existence means that for any given f, there exists a solution which satisfies the discrete problem. The uniqueness means that the solution is unique. Put it in another way, if $f = 0$, then the solution must be zero as well. Notice that for this discrete problem, both the existence and the uniqueness means that the matrix $\bar{A}$ is nonsingular. So we only need to prove one of them. Let us verify the uniqueness.

If $f = 0$, let v_h equals u_h and q_h equals p_h, then the second term in the first equation equals zero, and it implies

$$p_h = 0.$$

Then for each $q_h \in Q_h$,

$$(\mathrm{div}q_h, u_h) = 0.$$

Since Range$(Q_h) = V_h$, by choosing q_h such that $u_h = \mathrm{div}q_h$,

$$u_h = 0.$$

So we have the uniqueness of the solution. The uniqueness implies that the discrete system is nonsingular, therefore the existness of the solution. □

Next, we use the commutative diagram to give error estimate for the solution.

Lemma 87. *If* $p \in H^2(\Omega)$*, the following relationships hold:*

(12.5)
$$\mathrm{div}p_h = \mathrm{div}\Pi_h^{\mathrm{div}} p = \Pi_h^0 \mathrm{div}p = -\Pi_h^0 f,$$

(12.6)
$$(p - p_h, \Pi_h^{\mathrm{div}} p - p_h) = 0,$$

(12.7)
$$\|\mathrm{div}(p - p_h)\|_0 = \|\mathrm{div}p - \Pi_h^0 \mathrm{div}p\|_0 \lesssim h|\mathrm{div}p|_1,$$

and

(12.8)
$$\|p - p_h\|_0 \le \|p - \Pi_h^{\mathrm{div}} p\|_0 \lesssim h|p|_1$$

Proof. Since $p_h \in Q_h \subset H^{\rm div}(\Omega)$,

$$(\mathrm{div} p_h, v_h) = -(f, v_h) = (\mathrm{div} p, v_h),$$

we have

$$\mathrm{div} p_h = -\Pi_h^0 f = \Pi_h^0 \mathrm{div} p.$$

By the commutative property,

$$\mathrm{div} \Pi_h^{\rm div} p = \Pi_h^0 \mathrm{div} p.$$

Since $\Pi_h^0 q = q,\ \forall q|_K \in P_0(K)^2$, Bramble-Hilbert lemma tells

$$\|\mathrm{div}(p - p_h)\|_0 = \|\mathrm{div} p - \Pi_h^0 \mathrm{div} p\|_0 \lesssim h|\mathrm{div} p|_1.$$

Next, we prove the second identity. Since

$$(p - p_h, \Pi_h^{\rm div} p - p_h) + (\mathrm{div}(\Pi_h^{\rm div} p - p_h), u_h) = 0,$$

and $\mathrm{div}(\Pi_h^{\rm div} p - p_h) = 0$, the second term equals zero, so we have

$$(p - p_h, \Pi_h^{\rm div} p - p_h) = 0.$$

Then

$$\|p - p_h\|^2 = (p - p_h, p - \Pi_h^{\rm div} p) + (p - p_h, \Pi_h^{\rm div} p - p_h) = (p - p_h, p - \Pi_h^{\rm div} p) \lesssim \|p - p_h\|_0 \|p - \Pi_h^{\rm div} p\|_0.$$

Since $\Pi_h^{\rm div} q = q,\ \forall q|_K \in P_0(K)^2$, Bramble-Hilbert lemma tells

$$\|p - p_h\|_0 \lesssim \|p - \Pi_h^{\rm div} p\|_0 \lesssim h|p|_1.$$

□

12.2 Basic functional analysis results

First we consider a m by n matrix $B : \mathbb{R}^n \mapsto \mathbb{R}^m$ where $B = (b_1, \cdots, b_m)^T$ and $b_i \in \mathbb{R}^n$, $1 \le i \le n$. Denote the column space of B^T by $\mathrm{range}(B^T)$ and the null space of B by $\mathrm{null}(B)$ as below

$$\mathrm{range}(B^T) = \mathrm{span}\{b_1, \cdots, b_m\},$$

$$\mathrm{null}(B) = \{v \in \mathbb{R}^n : Bv = 0\}.$$

It is known that $\mathrm{range}(B^T) = \mathrm{null}(B)^\perp$.

Then the following statements are equivalent.

Lemma 88. *The following properties are equivalent:*

(1) The inf-sup condition holds:

$$\inf_{q\in\mathbb{R}^m} \sup_{v\in\mathbb{R}^n} \frac{q^T B v}{\|v\|\|q\|} \equiv \beta > 0; \qquad (12.9)$$

(2) The matrix B^T is an isomorphism from $\mathbb{R}^m$ onto $\mathrm{range}(B^T)$ *and*

$$\|B^T q\| \ge \beta \|q\| \quad \forall\ q \in \mathbb{R}^m; \qquad (12.10)$$

(3) The matrix B is an isomorphism from $\mathbb{R}^n/\text{null}(B)$ onto $\mathbb{R}^m$ and

$$\|Bv\| \geq \beta\|v\| \quad \forall\ v \in \text{null}(B)^\perp. \tag{12.11}$$

Proof. (1 ⇔ 2): The inequality (12.9) implies that

$$\sup_{v\in\mathbb{R}^n} \frac{q^T Bv}{\|v\|} \geq \beta\|q\|,\ \forall q \in \mathbb{R}^m.$$

The fact that $\|B^T q\| = \sup_{v\in\mathbb{R}^n} \frac{q^T Bv}{\|v\|}$ gives (12.10) and

$$\text{rank}(B^T) = m.$$

Thus, B^T is an isomorphism from $\mathbb{R}^m$ to range(B^T), which completes the proof for the equivalence between (1) and (2).

(2 ⇒ 3): For any $v \in \mathbb{R}^n/\text{null}(B)$,

$$Bv = 0 \Rightarrow v = 0.$$

Thus, B is an isomorphism from $\text{null}(B)^\perp$ onto $\mathbb{R}^m$. For any $v \in \text{null}(B)^\perp$,

$$\|Bv\| = \sup_{q\neq 0} \frac{(Bv, q)}{\|q\|} \geq \beta \sup_{B^T q\neq 0} \frac{(v, B^T q)}{\|B^T q\|} = \beta\|v\|.$$

(3 ⇒ 2):

$$\|B^T q\| = \sup_{v\neq 0} \frac{(B^T q, v)}{\|v\|} \geq \beta \sup_{Bv\neq 0} \frac{(q, Bv)}{\|Bv\|} = \beta\|q\|.$$

□

In the following, we consider the case that B is an operator. We first give a few well known theorems from functional analysis. We details on these results, we refer to Yosida [45].

Theorem 86 (Open mapping theorem). *If V and Q are Banach spaces and $B : V \mapsto Q$ is continuous and surjective, then B is an open map; namely if U is an open set in V, then $B(U)$ is an open set in Q.*

Denote

$$Z = \{v \in V :\ b(v, q) = 0 \quad \forall\ q \in Q\} = \text{Ker}(B).$$

Given a subspace $W \subset V$, we define its polar sets and $W^0 \subset V$ as follows

$$W^0 = \{f \in V' :\ \langle f, v\rangle = 0 \quad \forall\ v \in W\}.$$

Namely

$$W^0 = \text{Ker}(\imath_W')$$

where $\imath_W : W \mapsto V$ is the natural inclusion operator. Similarly, for a given subspace $F \subset V'$, we define its polar set ${}^0F \subset V'$

$${}^0F = \{x \in V :\ \langle f, v\rangle = 0 \quad \forall\ f \in F\}.$$

Namely

$${}^0F = \text{Ker}(\imath_F)$$

where $\imath_F : F \mapsto V'$ is the natural inclusion operator.

Theorem 87 (Closed range theorem). *Let V and Q be Banach spaces, and T a closed linear operator defined in V into Q such that domain of T, $D(T)$ is dense in V. The following propositions are all equivalent:*

- *$R(T)$ is closed in Q,*
- *$R(T')$ is closed in V',*
- *$R(T) = {}^0N(T') := \{q \in Q, \langle q', q\rangle = 0,$ for all $q' \in N(T')\}$,*
- *$R(T') = N(T)^0 := \{v' \in V', \langle v', v\rangle = 0,$ for all $v \in N(T)\}$.*

We also introduce the concept of **quotient space**. Given a Banach space V, and a closed subspace M of V, the quotient space V/M is defined as the set of equivalent classes $[v] := \{q \in V | q - v \in M\}$. Then V/M is also a Banach space with respect to the quotient norm

$$\|v\|_{V/M} := \inf_{y\in M} \|v - q\|_V.$$

For the case that $V = \mathbb{R}^n$ and B is a matrix,

$$Z = \text{null}(B).$$

Since $\text{range}(B^T) = \text{null}(B)^\perp$, we know that

$$Z^0 = \text{range}(B^T).$$

Next, we introduce a theorem for any operator B as a generalization of Lemma 88 for a matrix B.

Lemma 89. *The following properties are equivalent:*

1. *The inf-sup condition holds:*

$$\inf_{q\in Q} \sup_{v\in V} \frac{b(v,q)}{\|v\|_V \|q\|_Q} \equiv \beta > 0; \quad (12.12)$$

2. *The operator B' is an isomorphism from Q onto Z^0 and*

$$\|B'q\|_{V'} \geq \beta \|q\|_Q \quad \forall\, q \in Q; \quad (12.13)$$

3. *The operator B is an isomorphism from V/Z onto Q' and*

$$\|Bv\|_{Q'} \geq \beta \|v\|_{V/Z} \quad \forall\, v \in V. \quad (12.14)$$

or equivalently, when V is an Hilbert space, the operator B is an isomorphism from $Z^\perp$ onto Q' and

$$\|Bv\|_{Q'} \geq \beta \|v\|_V \quad \forall\, v \in Z^\perp. \quad (12.15)$$

Proof.
(1 ⇔ 2): Clearly the inequality (12.12) and (12.20) are equivalent. Hence (2) implies (1). Then we only need to prove that (12.20) implies that B' is an isomorphism from Q to Z^0. Since B' is one to one, we know that B' is isomorphism from Q to its range $R(B')$. Thus $R(B')$ is closed in V'. Now apply the Closed Range Theorem of Banach, we obtain that

$$\text{range}(B') = (\ker(B))^0 = Z^0.$$

This means that (1) implies (2). Thus we have proved that (1) and (2) are equivalent.

(2 ⇔ 3): The main thing is to establish that Z^0 can be isometrically identified with $(V/Z)'$. Indeed, with each $g \in (V/Z)'$, we associate the element $\tilde{g} \in V'$ by:

$$\langle \tilde{g}, v\rangle := \langle g, [v]\rangle \quad \forall v \in V$$

where $[v]$ is the equivalent class of v in V/Z. Obviously, $\tilde{g} \in Z^0$ and it is easy to check that the above correspondence is an isometric bijection.

We know that if an operator is invertible, then its adjoint is also invertible. Thus B' is an isomorphism from Q to Z^0 and (12.20) holds if and only if B is an isomorphism from V/Z to Q' and (12.21) holds.

In case that V is an Hilbert space, we only need to show that Z^0 can be isometrically identified with $(Z^\perp)'$, which follows from a similar argument with the Banach space case. □

12.3 Babuska Theory

The Babuska theory we will now describe has two parts. The first part concerns the well-posedness of a general linear variational problem and the second part concerns the approximation property of the discretization of the varitional problem.

Let V and Q be two Hilbert spaces, with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$ respectively. Let $b(\cdot,\cdot) : V\times Q \mapsto \mathbb{R}$ be a continuous bilinear form

$$b(u, q) \leq \|b\|\ \|u\|_V\|q\|_Q. \tag{12.16}$$

Consider the following variational problem: Find $u \in V$ such that

$$b(u, q) = \langle g, q\rangle, \quad \forall q \in Q, \tag{12.17}$$

where $g \in Q'$ (the space of continuous linear functionals on Q) and $\langle\cdot,\cdot\rangle$ is the usual pairing between Q' and Q.

By Riesz representation theorem, there exists $\tilde{g} \in Q$ such that

$$(\tilde{g}, q) = \langle g, q\rangle, \quad \forall q \in Q.$$

And

$$\|\tilde{g}\|_Q = \sup_{q\in Q}\frac{(\tilde{g}, q)}{\|q\|_Q} = \sup_{q\in Q}\frac{\langle g, q\rangle}{\|q\|_Q} = \|g\|_{Q'}.$$

Then the above problem becomes: Find $u \in V$ such that

$$b(u, q) = (\tilde{g}, q), \quad \forall q \in Q, \tag{12.18}$$

where $\tilde{g} \in Q$. In the rest of this chapter, we replace $\tilde{g}$ in (12.18) by g.

Denote

$$K := \{u \in V : b(u, q) = 0,\ \forall q \in Q\}.$$

According to Lemma 89, the inf-sup condition

$$\inf_{q\in Q}\sup_{v\in V}\frac{b(v, q)}{\|v\|_V\|q\|_Q} \equiv \beta > 0 \tag{12.19}$$

means

1. For any $f \in K^\perp$, there exists a unique $p \in Q$ such that

$$b(v, p) = (f, v),\ \forall v \in V.$$

$$\|f\|_V \geq \beta\|p\|_Q; \tag{12.20}$$

2. For any $g \in Q$, there exists a unique $u \in K^\perp$ such that

$$b(u, q) = (g, q),\ \forall q \in Q.$$

$$\|g\|_Q \geq \beta\|u\|_V. \tag{12.21}$$

The first part of Babuska's theory concerns the well-posedness (existence, uniquess and continuous data dependence) of the equation (12.18). For existence, the above analysis gives that if

$$\beta_1 \equiv \inf_{q\in Q}\sup_{v\in V}\frac{b(v, q)}{\|v\|_V\|q\|_Q} > 0, \tag{12.22}$$

there exists a unique solution $u \in K^{\perp} \subset V$.

The uniqueness means that if

$$b(u, q) = 0 \; \forall q \in Q,$$

$u = 0$ is resulted. Under the inf-sup condition

$$\beta_2 \equiv \inf_{v \in V} \sup_{q \in Q} \frac{b(v, q)}{\|v\|_V \|q\|_Q} > 0, \tag{12.23}$$

we know that u has to be zero such that

$$b(u, q) = 0, \; \forall q \in Q.$$

Thus it proves the uniqueness.

The above arguments lead to the first part of the Babuska theory concerns the well-posedness of the general variational problem.

Theorem 88. *The variational problem* (12.18) *is well-posed if and only if both inf-sup conditions* (12.22) *and* (12.23) *are satisfied. Furthermore when both* (12.22) *and* (12.23) *are satisfied, then*

$$\beta_1 = \beta_2$$

and the solution $u \in V$ satisfies

$$\|u\|_V \leq \beta^{-1} \|g\|_{V'}.$$

Exercise 1. Give a complete proof of Theorem 88.

Discrete Problem

The discrete variational problem is as follows: find $u_h \in U_h$ such that

$$b(u_h, q_h) = (g_h, q_h), \quad \forall q_h \in Q_h.$$

Similarly,

- Existence of solution:
$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} = c_0^h > 0$$
- Uniqueness of solution:
$$\inf_{v_h \in V_h} \sup_{q_h \in Q_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} = c_0^h > 0.$$

In the special case $Q_h \subset Q$ and $V_h \subset V$. Suppose $b(\cdot, \cdot)$ is continuous and the inf-sup conditions hold. Then

$$\|u - u_h\| \leq \frac{M_h}{c_0^h} \inf_{v_h \in V_h} \|u - v_h\|,$$

i.e. we have best approximation property. It means that if $c_0^h \geq \alpha_0 > 0, \quad \forall h$, the error is of optimal order.

12.4 Lax-Milgram Lemma

A useful special case of Babuska theorem is the well-known Lax-Milgram Lemma.

Theorem 89 (Lax-Milgram Lemma). *If $Q = V$ and the bilinear form b is coecive:*

$$b(v, v) \geq \beta \|v\|^2, \quad v \in V \tag{12.24}$$

for some $\beta > 0$, then the variational problem (12.18) *is well-posed.*

12.5 Discrete Petrov-Galerkin variational problems and quasi optimal approximation property

Let $V_h \subset V$ and $Q_h \subset Q$ be two nontrivial subspaces of V and Q respectively. We consider the following variational problem: Find $u_h \in V_h$ such that

$$b(u_h, q_h) = (g, q_h), \quad \forall q_h \in Q_h. \tag{12.25}$$

The solution u_h of this problem is often known as the Galerkin (or Petrov–Galerkin) approximation of u. Usually in applications V_h and Q_h are finite dimensional and the subscript h is related to certain discretization parameters (such as grid size and polynomial degree). According to (12.22) and (12.23) we have that the problem (12.25) is uniquely solvable if and only if the following conditions hold:

$$\inf_{u_h \in V_h} \sup_{q_h \in Q_h} \frac{b(u_h, q_h)}{\|u_h\|_V \|q_h\|_Q} = \inf_{q_h \in Q_h} \sup_{u_h \in V_h} \frac{b(u_h, q_h)}{\|u_h\|_V \|q_h\|_Q} = \beta_h > 0. \tag{12.26}$$

If V_h and Q_h are finite dimensional the above two conditions are reduced to one. A fundamental result for Galerkin approximation is as follows.

Theorem 90. *If the discrete variational problem (12.25) is wellposed, then*

$$\|u - u_h\|_V \le \left(\frac{M}{\beta_h} + 1\right) \inf_{v_h \in V_h} \|u - v_h\|_V.$$

Proof. We define $P_h : V \mapsto V_h$ s.t.

$$b(P_h u, q_h) = b(u, q_h), \forall q_h \in Q_h.$$

Then $P_h^2 = P_h$. By assumption,

$$\beta \|P_h u\|_V \le \sup_{q_h \in Q_h} \frac{b(P_h u, q_h)}{\|q_h\|_Q} = \sup_{q_h \in Q_h} \frac{b(u, q_h)}{\|q_h\|_Q}.$$

Therefore, $\|P_h u\|_V \le \frac{M}{\beta_h} \|u\|_V, \quad \forall v \in V$, i.e. $\|P_h\| \le M/\beta_h$. Finally, we have

$$\begin{aligned}
\|u - u_h\|_V &= \|u - P_h u\|_V \\
&= \|(I - P_h)(u - v_h)\| \le \|I - P_h\| \|u - v_h\|_V \\
&\le \left(1 + \frac{M}{\beta_h}\right) \|u - v_h\|_V
\end{aligned}$$

for any $v_h \in V_h$. This finishes our proof. □

In case $\beta_h > \beta_0 > 0$, then the constant $1 + M/\beta_h$ is bounded above independent of h. This corresponds to the case of quasi-optimal approximation.

In case that V_h is Hilbert space, then $\|I - P_h\| = \|P_h\|$ (Xu and Zikatanov (2002)). We will prove this result later in this section.

Theorem 91. *Let (12.16), (12.22), (12.23) and (12.26) hold. Then*

$$\|u - u_h\|_V \le \frac{\|b\|}{\beta_h} \inf_{v_h \in V_h} \|u - v_h\|_V. \tag{12.27}$$

Proof. Consider the mapping $P_h : V \mapsto V_h$ defined as $P_h u = u_h$. Using the fact that under the conditions of the theorem the problem (12.25) has a unique solution it is easy to see that this mapping is linear and idempotent, namely $P_h^2 = P_h$. The new twist in our proof is the identity

$$\|P_h\|_{\mathcal{L}(V,V)} = \|I - P_h\|_{\mathcal{L}(V,V)}, \tag{12.28}$$

which can be traced back to T. Kato [**?**] (see also Lemma 90 below). Applying this identity we get

$$\begin{aligned}\|u - u_h\|_V &= \|(I - P_h)(u - v_h)\|_V \le \|I - P_h\|_{\mathcal{L}(V,V)}\|u - v_h\|_V \\ &= \|P_h\|_{\mathcal{L}(V,V)}\|u - v_h\|_V,\end{aligned}$$

where $v_h \in V_h$ is arbitrary. By (12.26) and (12.16) we get

$$\|P_h u\|_V \le \frac{1}{\beta_h} \sup_{v_h \in V_h} \frac{b(u_h, v_h)}{\|v_h\|_Q} \le \frac{\|b\|}{\beta_h}\|u\|_V,$$

and the desired estimate (12.27) follows. □

Remark 27. The original theory of Babuska is slightly weaker than (12.27):

$$\|u - u_h\|_V \le \left(1 + \frac{\|b\|}{\beta_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_V.$$

The constant 1 was removed by Xu and Zikatanov (2002).

Theorem 92. *For a well-posed continuous problem* (12.18)*, if the discrete problem* (12.25) *has the following approximation property*

$$\|u - u_h\|_V \le C_h \inf_{v_h \in V_h} \|u - v_h\|_V \tag{12.29}$$

that holds for a constant C_h. Then the discrete inf-sup condition (12.26) *holds with*

$$\beta_h \ge \frac{\beta}{1 + C_h}$$

and in case that V is Hilbert space

$$\beta_h \ge \frac{\beta}{C_h}$$

Proof. Let $u_h = P_h u$, then by taking $v_h = 0$ in (12.29), we have

$$\|(I - P_h)u\|_V \le C_h\|u\|_V.$$

Thus

$$\|P_h u\|_V \le (1 + C_h)\|u\|_V.$$

and in case that V is Hilbert space

$$\|P_h u\|_V \le C_h\|u\|_V.$$

By the continuous inf-sup condition, we have, for any $q_h \in Q_h \subset Q$,

$$\begin{aligned}\beta\|q_h\|_Q &\le \sup_{v \in V_h} \frac{b(v, q_h)}{\|v\|_V} \\ &= \sup_{v \in V_h} \frac{b(P_h v, q_h)}{\|v\|_V} \\ &\le (1 + C_h) \sup_{v \in V_h} \frac{b(P_h v, q_h)}{\|P_h v\|_V} \\ &= (1 + C_h) \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V}.\end{aligned}$$

□

Theorem 93. *For a well-posed continuous problem* (12.18)*, the discrete problem* (12.25) *is uniformly well-posed if any only if it provides uniform quasi-optimal approximation, namely*

$$\|u - u_h\|_V \leq C \inf_{v_h \in V_h} \|u - v_h\|_V \tag{12.30}$$

that holds for a constant C independent of h.

Proof. This theorem is a consequence of Theorem 90 and Theorem 92.

Suppose that the discrete problem (12.25) is uniformly wellposed. Then it satisfies the inf-sup condition (12.26) with β_h independent of h. By Theorem 90, we have the quasi-optimal approximation with C_h independent of h. Then it satisfies uniform quasi-optimal approximation property.

If we assume uniform quasi-optimal approximation property, then (12.29) holds with C_h independent of h. Then by Theorem 92, we have the uniform inf-sup condition. □

Lax equivalence theorem for conforming Petro-Galerkin method

We focus on the following closely-related concepts. We denote by u_h the solution to Problem (12.25).

- Consistency:
$$\inf_{v_h \in V_h} \|u - v_h\|_V \to 0, \quad \text{as } h \to 0.$$
- Convergence:
$$\|u - u_h\|_V \to 0, \quad \text{as } h \to 0.$$
- Stability: discrete inf-sup condition
$$\inf_{u_h \in V_h} \sup_{q_h \in Q_h} \frac{b(u_h, q_h)}{\|u_h\|_V \|q_h\|_Q} = \inf_{q_h \in Q_h} \sup_{u_h \in V_h} \frac{b(u_h, q_h)}{\|u_h\|_V \|q_h\|_Q} = \beta > 0, \tag{12.31}$$
or, equivalently, quasi optimal approximation property (12.30).

Theorem 94 (Lax equivalence theorem for conforming Petrov Galerkin methods). *For a consistent conforming Petro Galerkin method, convergence is equivalent to stability.*

Proof. By the quasi optimal approximation property (12.30), we can easily prove convergence assuming stability.

Given convergence, we know that for any given $u \in V$,

$$\lim_h \|u - P_h u\| = 0.$$

Then $\|u - P_h u\| \leq C(u)$, i.e. $\|u - P_h u\|$ is bounded by a constant that only depends on u. By uniform boundedness principle, we get the conclusion

$$\|P_h\| \leq 1 + C.$$

Note that the upper bound is independent of u. □

We have the following facts:

1. Given consistency, convergence and stability are equivalent.
2. Convergence directly implies consistency and thus stability can also be proved.
3. Given stability, consistency implies convergence.

An identity for nontrivial idempotent operator

For completeness, we shall now describe a general result related to the identity (12.28) and include a (new) proof. This result can be traced back to Kato [**?**] and a more general result can be found in Zikatanov [**?**].

Lemma 90. *Let H be a Hilbert space with a norm $\|\cdot\|_H$ and inner product $(\cdot,\cdot)_H$. Let $P : H \mapsto H$ be an idempotent, such that $0 \neq P^2 = P \neq I$. Then the following identity holds*

$$\|P\|_{\mathcal{L}(H,H)} = \|I - P\|_{\mathcal{L}(H,H)}. \tag{12.32}$$

Proof. We first prove the theorem when $\dim H = 2$. Then both P and $I - P$ have to be rank 1, namely $Pv = (b, v)_H a$ and $(I - P)v = (d, v)_H c$ for some fixed nonzero $a, b, c, d \in H$ satisfying $(a, b)_H = (c, d)_H = 1$ and for all $v \in H$ we also have

$$v = Pv + (I - P)v = (b, v)_H \, a + (d, v)_H \, c.$$

A simple manipulation of the above identities yields that

$$\|a\|_H^2 \|b\|_H^2 = \|c\|_H^2 \|d\|_H^2 = 1 - (a, c)_H (b, d)_H.$$

The desired identity then follows because of the following obvious relations:

$$\|P^*P\|_{\mathcal{L}(H,H)} = \|a\|_H^2 \|b\|_H^2 \quad \text{and} \quad \|(I - P)^*(I - P)\|_{\mathcal{L}(H,H)} = \|c\|_H^2 \|d\|_H^2.$$

In general, for any given $x \in H$ such that $\|x\| = 1$, we consider a subspace $X = \mathrm{span}\{x,\ Px\}$. We note that X is invariant with respect to P and $I - P$. If $\dim X = 1$, then we must have $(I - P)x = 0$. If $\dim X = 2$, we have from two dimensional result just proved, $\|(I - P)x\|_X \le \|P\|_X$. In any case, we have

$$\|(I - P)x\|_H = \|(I - P)x\|_X \le \|P\|_{\mathcal{L}(X,X)} \le \|P\|_{\mathcal{L}(H,H)},$$

which implies $\|I - P\|_{\mathcal{L}(H,H)} \le \|P\|_{\mathcal{L}(H,H)}$. Similarly $\|P\|_{\mathcal{L}(H,H)} \le \|I - P\|_{\mathcal{L}(H,H)}$. This completes the proof. □

12.6 Brezzi theory

The Brezzi theory we shall now describe concerns the well-posedness and the discretization of a general mixed variational problem as follows:

$$\text{Find } (u, p) \in V \times Q, \quad \begin{cases} a(u, v) + b(v, p) = (f, v) & \forall v \in V, \\ b(u, q) \qquad\qquad\;\; = (g, q) & \forall q \in Q. \end{cases} \tag{12.33}$$

Here V and Q are two Hilbert spaces, $f \in V$, $g \in Q$ and $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are two continuous bilinear forms

$$\begin{aligned} &a(\cdot,\cdot) : V \times V \mapsto \mathbb{R};\ a(u, v) \le \|a\|\|u\|_V\|v\|_V,\ \forall u \in V,\ \forall v \in V, \\ &b(\cdot,\cdot) : V \times Q \mapsto \mathbb{R};\ b(v, q) \le \|b\|\|v\|_V\|q\|_Q,\ \forall v \in V,\ \forall q \in Q, \end{aligned}$$

In principle, we can use the BabuVska theory to study the above mixed variational problem. Setting $\mathscr{B}((u, p), (v, q)) = a(u, v) + b(v, p) + b(u, q)$, then (12.33) is obviously equivalent to the following problem

$$\mathscr{B}((u, p), (v, q)) = \langle f, v\rangle + \langle g, q\rangle, \quad \forall (v, q) \in V \times Q. \tag{12.34}$$

Then, following Babuska theory, this mixed variational problem (12.33) is well-posed if and only if the following BB-conditions hold:

$$\inf_{(u,p)\in V\times Q}\sup_{(v,q)\in V\times Q}\frac{\mathcal{B}((u,p),(v,q))}{\|(u,p)\|_{V\times Q}\|(v,q)\|_{V\times Q}}= \inf_{(v,q)\in V\times Q}\sup_{(u,p)\in V\times Q}\frac{\mathcal{B}((u,p),(v,q))}{\|(u,p)\|_{V\times Q}\|(v,q)\|_{V\times Q}}\equiv\gamma>0, \tag{12.35}$$

where

$$\|(v,q)\|^2_{V\times Q}=\|v\|^2_V+\|q\|^2_Q,\quad \forall(v,q)\in V\times Q.$$

But the conditions in (12.35) are not very useful and they are very difficult to verify.

We shall now derive the Brezzi theory that gives more convenient necessary and sufficient conditions for the well-posedness of our mixed variational problem.

By the analysis in section 12.3, we can find a unique $u_1\in K^\perp$ such that

$$b(u_1,q)=(g,q),\ \forall q\in Q. \tag{12.36}$$

And

$$\|u_1\|_V\le(1+\epsilon)\|u_1\|_{V/K}\le(1+\epsilon)\beta^{-1}\|g\|_Q.$$

Let $u_0\in K$ satisfy

$$a(u_0,v)=(f,v)-a(u_1,v),\quad v\in K. \tag{12.37}$$

So we need this problem to be well-posed. By Babuska's theory, we need the following two inf-sup conditions:

$$\inf_{u\in K}\sup_{v\in K}\frac{a(u,v)}{\|u\|_V\|v\|_V}=\inf_{v\in K}\sup_{u\in K}\frac{a(u,v)}{\|u\|_V\|v\|_V}\equiv\alpha>0. \tag{12.38}$$

Then we have the following estimates

$$\|u_0\|\le\alpha^{-1}(\|f\|_V+\|a\|\|u_1\|)\le\alpha^{-1}(\|f\|_V+\|a\|\beta^{-1}(1+\epsilon)\|g\|_Q).$$

Let $u=u_0+u_1$. It remains to determine p. By (12.37), we have

$$a(u,v)=(f,v),\ \forall v\in K.$$

Then there exists $p\in Q$ such that

$$b(v,p)=(f,v)-a(u,v),\ \forall v\in V. \tag{12.39}$$

Moreover, we have the following estimate

$$\|p\|_Q\le\beta^{-1}(\|f\|_V+\|a\|\|u\|_V).$$

Therefore,

$$\begin{aligned}&\|u\|_V+\|p\|_Q\\ \le&\beta^{-1}\|f\|_V+(1+\beta^{-1}\|a\|)\|u\|_V\\ \le&\beta^{-1}\|f\|_V+(1+\beta^{-1}\|a\|)(\alpha^{-1}\|f\|_V+(1+\alpha^{-1}\|a\|)\beta^{-1}(1+\epsilon)\|g\|_Q)\\ \le&(\alpha^{-1}+\beta^{-1}+\alpha^{-1}\beta^{-1}\|a\|)\|f\|_V+(1+\beta^{-1}\|a\|)(1+\alpha^{-1}\|a\|)\beta^{-1}(1+\epsilon)\|g\|_Q.\end{aligned}$$

Note that the coercive property of the bilinear form $a(\cdot,\cdot)$, namely

$$a(u,u)\ge C\|u\|^2_V,$$

indicates the inf-sup condition (12.38).

Consider the operator $\mathcal{L}$ defined by $\langle \mathcal{L}(u,p),(v,q)\rangle := \mathcal{B}((u,p),(v,q))$. We have

$$\|\mathcal{L}^{-1}\|_{V\times Q\mapsto V\times Q} \le \max\{\alpha^{-1}+\beta^{-1}+\alpha^{-1}\beta^{-1}\|a\|,(1+\beta^{-1}\|a\|)(1+\alpha^{-1}\|a\|)\beta^{-1}(1+\epsilon)\}.$$

Since $\epsilon > 0$ is arbitrary, we actually have

$$\|\mathcal{L}^{-1}\|_{V\times Q\mapsto V\times Q} \le \max\{\alpha^{-1}+\beta^{-1}+\alpha^{-1}\beta^{-1}\|a\|,(1+\beta^{-1}\|a\|)(1+\alpha^{-1}\|a\|)\beta^{-1}\}.$$

With these and some additional simple arguments, we obtain the following Theorem of Brezzi.

Theorem 95 (Brezzi [?]). *The variational problem (12.33) is well posed if and only if the following BB-conditions hold*

$$\inf_{u\in K}\sup_{v\in K}\frac{a(u,v)}{\|u\|_V\|v\|_V} = \inf_{v\in K}\sup_{u\in K}\frac{a(u,v)}{\|u\|_V\|v\|_V} \equiv \alpha > 0, \tag{12.40}$$

where $K = \{v\in V : b(v,q)=0,$ *for all* $q\in Q\}$, *and*

$$\inf_{q\in Q}\sup_{v\in V}\frac{b(v,q)}{\|v\|_V\|q\|_Q} \equiv \beta > 0. \tag{12.41}$$

Furthermore, under the conditions of (12.41) *and* (12.40)*, the unique solution* $(u,p)\in V\times Q$ *of (12.33) satisfies*

$$\|(u,p)\|_{V\times Q} \le \mathbf{K}(\alpha^{-1},\beta^{-1},\|a\|)\|(f,g)\|_{V\times Q}, \tag{12.42}$$

where $\mathbf{K}(\cdot,\cdot,\cdot)$ *is a function which is increasing in each variable.*

Exercise 2. Give a complete proof of Theorem 95.

Example 8. We recall the mixed formulation (12.67) for Poisson problem with

$$a(u,v) = (\alpha^{-1}u,v), \quad b(u,q) = (\mathrm{div}u,q).$$

And use the Brezzi theory to prove the existence of solution for this problem. By Brezzi's theorem, we only need to verify two inf-sup conditions. For each $u\in V$ such that $\mathrm{div}u = 0$, we have

$$a(u,u) = (\alpha^{-1}u,u) \ge \alpha_1^{-1}\|u\|^2 = \alpha_1^{-1}\|u\|^2_{H(\mathrm{div})}.$$

For the inf-sup condition concerning $b(\cdot,\cdot)$, for any $q\in Q$, consider the auxiliary problem

$$\begin{aligned} -\Delta\phi &= q \quad \text{In } \Omega \\ \phi &= 0 \quad \text{on } \Gamma \end{aligned}$$

Take $u = \nabla\phi \in Q$, it is then easy to see that

$$(\mathrm{div}u,q) = \|q\|^2.$$

Since $\|u\|_V^2 = \|u\|_0^2 + \|\mathrm{div}u\|_0^2 = \|\nabla\phi\|_0^2 + \|\Delta\phi\|_0^2 \le C\|q\|_0^2$,

$$\sup_{v\in V}\frac{b(v,q)}{\|v\|_V} \ge \beta\|q\|_Q.$$

We shall now briefly discuss the Galerkin approximation for (12.33). We consider two nontrivial finite dimensional subspaces $V_h\subset V$ and $Q_h\subset Q$ and the following variational problem:

$$\begin{cases} a(u_h,v_h)+b(v_h,p_h) = (f,v_h) & \forall v_h\in V_h,\\ b(u_h,q_h) = (g,q_h) & \forall q_h\in Q_h. \end{cases} \tag{12.43}$$

Theorem 96. *Let* $V_{h,0} = \{v_h \in V_h : b(v_h, q_h) = 0, \quad \forall q \in Q_h\}$ *and assume that the following BB-conditions hold*

$$\inf_{u_h \in V_{h,0}} \sup_{v_h \in V_{h,0}} \frac{a(u_h, v_h)}{\|u_h\|_V \|v_h\|_V} \equiv \alpha_h > 0, \tag{12.44}$$

and

$$\inf_{q_h \in Q_h} \sup_{v_h \in V_h} \frac{b(v_h, q_h)}{\|v_h\|_V \|q_h\|_Q} \equiv \beta_h > 0. \tag{12.45}$$

Then the discrete problem (12.43) is well-posed and

$$\|(u - u_h, p - p_h)\|_{V \times Q} \leq (\|a\| + \|b\|)\mathbf{K}(\alpha_h^{-1}, \beta_h^{-1}, \|a\|) \inf_{(v_h, q_h) \in V_h \times Q_h} \|(u - v_h, p - q_h)\|_{V \times Q}.$$

Furthermore if $\alpha_h \geq \alpha_0$ *and* $\beta_h \geq \beta_0$ *for some positive constants* α_0 *and* β_0*, then*

$$\|(u - u_h, p - p_h)\|_{V \times Q} \leq (\|a\| + \|b\|)\mathbf{K}(\alpha_0^{-1}, \beta_0^{-1}, \|a\|) \inf_{(v_h, q_h) \in V_h \times Q_h} \|(u - v_h, p - q_h)\|_{V \times Q}.$$

We would like to remark that the above approximation result is a direct consequence of Theorem 91 andTheorem 95 and the obvious estimate that $\|\mathbb{B}\| \leq \|a\| + \|b\|$. In some of the existing works, another approach in proving Theorem 96 is considered (see [**?**], [**?**])) and some additional arguments are needed, first to establish estimate for $u - u_h$ and then for $p - p_h$. This more refined analysis can be interesting in some applications (for example, when the BB-conditions are not uniformly satisfied), but it may not be necessary in general.

In most case, condition (12.44) is easy to be verified. Hence, the most critical is the inf-sup condition for $b(\cdot, \cdot)$ in most case. In next part, we will give some important inequalities that may be used to prove the inf-sup condition.

12.7 More on Brezzi theory: a generalization

Find $(u, p) \in V \times Q$, s.t.

$$\begin{cases} a(u, v) + b(v, p) = (f, v), & \forall v \in V, \\ b(u, q) - c(p, q) = (g, q), & \forall q \in Q. \end{cases} \tag{12.46}$$

Introduce the following bilinear form

$$L((u, p), (v, q)) = a(u, v) + b(v, p) - b(u, q) + c(p, q).$$

If

$$\begin{pmatrix} A & B' \\ B & 0 \end{pmatrix}$$

is an isomorphism, what is the right condition for C such that

$$\begin{pmatrix} A & B' \\ B & -C \end{pmatrix}$$

is also an isomorphism?

Define

$$K := \{v \in V : b(v, q) = 0, \ \forall q \in Q\}.$$
$$H := \{q \in Q : b(v, q) = 0, \ \forall v \in V\}.$$

We assume the following:

(12.47) $$a(\cdot,\cdot) \text{ and } c(\cdot,\cdot) \text{ are symmetric and positive.}$$

(12.48) $$a(u,v) \le C_1\|u\|_V\|v\|_V, \quad b(v,q) \le C_3\|v\|_V\|q\|_Q, \quad c(p,q) \le C_2\|p\|_Q\|q\|_Q.$$

(12.49) $$a(v,v) \ge M_1\|v\|_V^2, \ \ \forall v \in K, \quad c(q,q) \ge M_2\|q\|_Q^2, \ \ \forall q \in H.$$

(12.50) $$\inf_{q\in H^\perp}\sup_{v\in V}\frac{b(v,q)}{\|v\|_V\|q\|_Q} = \inf_{v\in K^\perp}\sup_{q\in Q}\frac{b(v,q)}{\|v\|_V\|q\|_Q} \ge \beta.$$

Based on these assumptions, we prove the wellposedness of the saddle point problem by verifying the inf-sup condition of $L((u,p),(v,q))$.

Define the semi-norms:
$$|v|_a^2 := a(v,v) \quad |q|_c^2 := c(q,q).$$
We have
$$a(u,v) \le |u|_a|v|_a \quad c(p,q) \le |p|_c|q|_c.$$
Since $(a(v,w))^2 \le a(v,v)a(w,w), \forall v,w \in V$,
$$a(v,w)^2 \le C_1|v|_a^2\ \|w\|_V^2.$$
Thus,

(12.51) $$(\sup\frac{a(v,w)}{\|w\|_V})^2 \le C_1|v|_a^2.$$

Similarly,

(12.52) $$\sup\frac{c(p,q)}{\|q\|_Q} \le C_2^{1/2}|p|_c.$$

Theorem 97. *Assume that $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are symmetric positive semi-definite and (12.48), (12.49) and (12.50) hold. Moreover, assume that $|u|_a := \sqrt{a(u,u)}$ and $|p|_c := \sqrt{c(p,p)}$ define semi norms. Then (12.46) is wellposed.*

Proof. Define the mapping $M : V\times Q \to V\times Q$ as $(u,p)\to(f,g)$. For any given $(u,p)\in V\times Q$, let $(f,g)\in V\times Q$ such that
$$\begin{cases} a(u,v)+b(v,p)=(f,v), & \forall v\in V,\\ b(u,q)-c(p,q)=(g,q), & \forall q\in Q.\end{cases}$$
In order to prove the wellposedness, it needs to prove that M is surjective and injective. Since $M'=M$, if M is injective, then M' is injective, meaning
$$\overline{R(M)} = Ker(M')^\perp = V\times Q.$$
If M is injective, namely M^{-1} is bounded. Then for any Cauchy sequence $M(u_n,p_n)$, since
$$\|(u_n,p_n)-(u_n,p_n)\|_{V\times Q} \le \|M^{-1}\|\|M(u_n,p_n)-M(u_n,p_n)\|,$$

$\|(u_n, p_n)\|$ is a Cauchy sequence in $V \times Q$. Thus, there exists $(u, p) \in V \times Q$ such that $(u_n, p_n) \to (u, p)$. Thus

$$M(u_n, p_n) \to M(u, p) \in \mathrm{R(M)}.$$

So $R(M) = \overline{R(M)}$. Thus, by $M' = M$, if M is injective, it is also surjective. Thus, it only remains to prove that M is injective, namely,

$$\|u\|_V + \|p\|_Q \leq C(\|f\|_V + \|g\|_Q). \tag{12.53}$$

By Lemma 89, (12.50) implies

$$\sup_{q\in Q} \frac{b(v,q)}{\|q\|_Q} \geq \beta\|v\|_V, \quad \forall v \in K^{\perp}.$$

For any given $(u, p) \in \mathbb{V} \times \mathbb{Q}$, we can do splitting for u, p as follows

$$u = u_0 + u_1, \quad u_0 \in K, u_1 \in K^{\perp}$$
$$p = p_0 + p_1, \quad p_0 \in H, p_1 \in H^{\perp}.$$

Similarly, we can split $f \in V$ and $g \in Q$ as

$$f = f_0 + f_1 \quad g = g_0 + g_1$$

with $f_0 \in K, f_1 \in K^{\perp}, g_0 \in H, g_1 \in H^{\perp}$. Thus,

$$(f, v) = (f_0, v_0) + (f_1, v_1) \quad (g, q) = (g_0, q_0) + (g_1, q_1).$$

Let the test function $v = u_0$ for the first equation (12.46), then

$$a(u_0, u_0) = (f, u_0) - a(u_1, u_0).$$

By (12.49),

$$\|u_0\|_V \lesssim \|f\|_V + \|u_1\|_V. \tag{12.54}$$

Let the test function $q = p_0$ for the second equation (12.46), then

$$c(p_0, p_0) = (g, p_0) - c(p_1, p_0).$$

By (12.49),

$$\|p_0\|_Q \lesssim \|g\|_Q + \|p_1\|_Q. \tag{12.55}$$

Since $b(u_1, q) = (g, q) - c(p, q), \ \forall q \in Q$, by (12.50),

$$\|u_1\|_V \leq \frac{1}{\beta} \sup_{q\in Q} \frac{b(u_1, q)}{\|q\|_Q} \lesssim \|g\|_Q + \|p\|_Q. \tag{12.56}$$

Combining (12.54), (12.55) and (12.56), we have

$$\|p_0\|_Q \lesssim \|g\|_Q + \|p_1\|_Q, \ \|u_1\|_V \lesssim \|g\|_Q + \|p_1\|_Q, \ \|u_0\|_V \lesssim \|f\|_V + \|g\|_Q + \|p_1\|_Q. \tag{12.57}$$

Taking $v = u$ in the first equation, $q = p$ in the second equation, and subtracting , we have

$$a(u, u) + c(p, p) = (f, u) - \langle g, p\rangle.$$

It follows from (12.51) and (12.52) that

(12.58) $$\frac{a(u,v)}{C_1\|v\|_V}+\frac{c(p,q)}{C_2\|q\|_Q}\le (f,u)-(g,p),\ \forall v\in V,\ q\in Q.$$

Since $b(v,p)=(f,v)-a(u,v),\ \forall v\in V$,

(12.59) $$(\sup_{v\in V}\frac{b(v,p)}{\|v\|_V})^2=(\sup_{v\in V}\frac{(f,v)-a(u,v)}{\|v\|_V})^2\le\|f\|_V^2+(\sup_{v\in V}\frac{a(u,v)}{\|v\|_V})^2\lesssim\|f\|_V^2+(f,u)+(g,p).$$

By (12.57) and Young inequality,

(12.60) $$\|p_1\|_Q^2\le\frac{1}{\beta^2}(\sup_{v\in V}\frac{b(v,p)}{\|v\|_V})^2\lesssim\|f\|_V^2+\|g\|_Q^2+\frac{1}{2}\|p_1\|_Q^2.$$

Thus, $\|p_1\|_Q\lesssim\|f\|_V+\|g\|_Q$. By (12.57), we have

$$\|u\|_V+\|p\|_Q\le C(\|f\|_V+\|g\|_Q),$$

and complete the proof.

□

Below is a constructive proof for this theorem. Before presenting the theorem, we first introduce two lemmas.

Lemma 91. *Given $\beta>0$, $\alpha\in\mathbb{R}^1$, the following inequality holds for any $x,y\in\mathbb{R}^1$*

$$x^2-t\alpha xy+t\beta y^2\ge\frac{1}{2}\left(x^2+\frac{\beta^2}{\alpha^2}y^2\right)$$

with $t=\beta/\alpha^2$.

Proof. Since $x^2-xz+z^2\ge(x^2+z^2)/2$, the proof follows by letting $z=\beta y/\alpha$. □

Lemma 92. *Given Hilbert space $\mathbb{V}$,*

$$\alpha\|u\|^2+\beta\|v\|^2\ge\frac{\alpha\beta}{\alpha+\beta}\|u+v\|^2,\quad\forall\alpha,\beta>0 \text{ and } u,v\in\mathbb{V}.$$

Proof. We just need to prove

$$(1+\alpha/\beta)\|u\|^2+(1+\beta/\alpha)\|v\|^2\ge\|u+v\|^2,$$

which follows from Young's inequality:

$$\frac{\alpha}{\beta}\|u\|^2+\frac{\beta}{\alpha}\|v\|^2\ge 2(u,v).$$

□

Proof. Due to the inf-sup conditions of $b(\cdot,\cdot)$, we can find $w\in V$ and $r\in Q$ s.t.

$$b(w,p_1)\ge\beta\|p_1\|_Q^2,\ \ \|w\|_V=\|p_1\|_Q\quad\text{and}\quad b(u_1,r)\ge\beta\|u_1\|_V^2,\ \ \|r\|_Q=\|u_1\|_V.$$

Take $v=u+\theta_1 w$, $q=p-\theta_2 r$. Then

$$
\begin{aligned}
& L((u,p),(v,q)) \\
=& a(u,u+\theta_1 w) + b(u+\theta_1 w, p) - b(u, p-\theta_2 r) + c(p, p-\theta_2 r) \\
=& a(u,u) + \theta_1(a(u,w) + b(w,p)) - \theta_2(c(p,r) - b(u,r)) + c(p,p) \\
\geq& a(u,u) - \theta_1 |u|_a |w|_a + \theta_1\beta\|p_1\|_Q^2 + \theta_2\beta\|u_1\|_V^2 - \theta_2 |p|_c |r|_c + c(p,p) \\
\geq& a(u,u) - \theta_1 C_1^{1/2} |u|_a \|w\|_V + \theta_1\beta\|p_1\|_Q^2 + \theta_2\beta\|u_1\|_V^2 - \theta_2 C_2^{1/2} |p|_c \|r\|_Q + c(p,p) \\
=& a(u,u) - \theta_1 C_1^{1/2} |u|_a \|p_1\|_Q + \theta_1\beta\|p_1\|_Q^2 + \theta_2\beta\|u_1\|_V^2 - \theta_2 C_2^{1/2} |p|_c \|u_1\|_V + c(p,p) \\
& \text{(use Lemma 91)} \\
\geq& \frac{1}{2}|u|_a^2 + \frac{\beta^2}{2C_2}\|u_1\|_V^2 + \frac{1}{2}|p|_c^2 + \frac{\beta^2}{2C_1}\|p_1\|_Q^2. \\
& \text{(use Lemma 91 and define } \gamma_1 = \beta^2/4C_1, \gamma_2 = \beta^2/4C_2) \\
=& \frac{1}{2}\left(|u|_a^2 + \gamma_2\|u_1\|_V^2 + |p|_c^2 + \gamma_1\|p_1\|_Q^2\right). \\
& \text{(with } 0 < t < \gamma_2, 0 < s < \gamma_1) \\
=& \frac{1}{2}\left(|u|_a^2 + (\gamma_2 - t)\|u_1\|_V^2 + t\|u_1\|_V^2 + |p|_c^2 + (\gamma_1 - s)\|p_1\|_Q^2 + s\|p_1\|_Q^2\right). \\
& \text{(by the boundedness of } a(\cdot,\cdot) \text{ and } c(\cdot,\cdot)) \\
\geq& \frac{1}{2}\left(|u|_a^2 + (\gamma_2 - t)\|u_1\|_V^2 + \frac{t}{C_1}|u_1|_a^2 + |p|_c^2 + (\gamma_1 - s)\|p_1\|_Q^2 + \frac{s}{C_2}|p_1|_c^2\right). \\
& \text{(by Lemma 92)} \\
\geq& \frac{1}{2}\left((\gamma_2 - t)\|u_1\|_V^2 + \frac{t}{C_1 + t}|u_0|_a^2 + (\gamma_1 - s)\|p_1\|_Q^2 + \frac{s}{C_2 + s}|p_0|_c^2\right). \\
& \text{by coercivity of } a(\cdot,\cdot), c(\cdot,\cdot) \\
\geq& \frac{1}{2}\left((\gamma_2 - t)\|u_1\|_V^2 + \frac{M_1 t}{C_1 + t}\|u_0\|_V^2 + (\gamma_1 - s)\|p_1\|_Q^2 + \frac{M_2 s}{C_2 + s}\|p_0\|_Q^2\right). \\
\geq& \lambda_1\|u\|_V^2 + \lambda_2\|p\|_Q^2,
\end{aligned}
$$

where

$$
\lambda_1 = \frac{1}{2}\min\left\{\gamma_2 - t, \frac{M_1 t}{C_1 + t}\right\}, \quad \lambda_2 = \frac{1}{2}\left\{\gamma_1 - s, \frac{M_2 s}{C_2 + s}\right\}.
$$

We can maximize the value of λ_1 since it is a function of t. By letting

$$
\gamma_2 - t = \frac{M_1 t}{C_1 + t},
$$

we find the following maximizer:

$$
t_{max} = \frac{-(M_1 + C_1 - \gamma_2) + \sqrt{(M_1 + C_1 - \gamma_2)^2 + 4C_1\gamma_2}}{2}.
$$

Then

$$
\begin{aligned}
\lambda_1(t_{max}) &= \gamma_2 - t_{max} \\
&= \frac{(M_1 + C_1 + \gamma_2) - \sqrt{(M_1 + C_1 - \gamma_2)^2 + 4C_1\gamma_2}}{2} \\
&= \frac{(M_1 + C_1 + \gamma_2) - \sqrt{(M_1 + C_1 + \gamma_2)^2 - 4\gamma_2 M_1}}{2} \\
&= \frac{2\gamma_2 M_1}{(M_1 + C_1 + \gamma_2) + \sqrt{(M_1 + C_1 + \gamma_2)^2 - 4\gamma_2 M_1}} \\
&\quad (\text{drop} - 4\gamma_2 M_1) \\
&\geq \frac{\gamma_2 M_1}{M_1 + C_1 + \gamma_2}.
\end{aligned}
$$

Similarly, we get

$$\lambda_2(s_{max}) \geq \frac{\gamma_1 M_2}{M_2 + C_2 + \gamma_1}.$$

Moreover,

$$\|v\|_V^2 + \|q\|_Q^2 \leq \left(1 + \frac{\beta^2}{C_2^2}\right)\|u\|_V^2 + \left(1 + \frac{\beta^2}{C_1^2}\right)\|p\|_Q^2.$$

Therefore, we have proved the wellposedness of (12.46). □

12.8 General settings revisited

We consider the following variational problem: Find $(u, p) \in \mathbb{V} \times \mathbb{Q}$ s.t.

(12.61)
$$\begin{cases} a(u, v) + b(v, p) & = \langle f, v \rangle, \quad \forall v \in \mathbb{V} \\ b(u, q) & = \langle g, q \rangle, \quad \forall q \in \mathbb{Q} \end{cases}$$

We know that this problem is well-posed if

(1) $a(u, v) \leq M_a \|u\|\|v\|$,
(2) $b(v, q) \leq M_b \|v\|\|q\|$,
(3)

(12.62)
$$\inf_{v \in \mathbb{V}} \sup_{q \in \mathbb{Q}} \frac{b(v, q)}{\|v\|_{\mathbb{V}} \|q\|_{\mathbb{Q}}} = \beta > 0.$$

(4) $a(v, v) \geq \alpha \|v\|_{\mathbb{V}}^2, \forall v \in \text{Ker}(B) = \{v \in \mathbb{V}, b(v, q) = 0, \forall q \in \mathbb{Q}\}$.

Similarly, we have operators A and B defined by

$$
\begin{aligned}
A : V \mapsto \mathbb{V}^*, &\text{ s.t. } \langle Au, v \rangle = a(u, v), \quad \forall u, v \in \mathbb{V} \\
B : V \mapsto \mathbb{Q}^*, &\text{ s.t. } \langle Bu, q \rangle = b(u, q), \quad \forall q \in \mathbb{Q}.
\end{aligned}
$$

Then the operator form follows:

(12.63)
$$\begin{cases} Au + B^* p & = f \\ Bu & = g. \end{cases} \text{ or, equivalently, } \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

To prove the wellposedness, the most critical part is the inf-sup condition for $b(\cdot, \cdot)$.

$$
\begin{aligned}
&B : \mathbb{V} \mapsto \mathbb{Q}^* \text{is surjective} \\
\Leftrightarrow &B^* : \mathbb{Q} \mapsto \mathbb{V}^* \text{is one-to-one} \\
\Leftrightarrow &Ker(B^*) = 0.
\end{aligned}
$$

We know this is true due to (12.62)

12.9 A mixed method for second order elliptic boundary value problems

We consider the following Dirichlet boundary value problem

$$\begin{aligned} -\text{div } \cdot (\alpha(x)\text{grad } p) &= f \quad \text{in } \Omega, \\ p &= 0 \quad \text{on } \Gamma_D \\ \frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_N. \end{aligned} \tag{12.64}$$

The variational problem of this problem is: Find $p \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$(\alpha\text{grad } p, \text{grad } q) = (f, q) \quad \forall q \in H^1_{0,\Gamma_D}(\Omega) \tag{12.65}$$

where

$$H^1_{0,\Gamma_D}(\Omega) = \{v \in H^1(\Omega) : p = 0 \text{ on } \Gamma_D\}.$$

Introducing the following intermediate variable:

$$u = \alpha\text{grad } p.$$

We have the following first order system

$$\begin{cases} \alpha^{-1}u - \text{grad } p = 0 \\ \text{div } u = -f \end{cases} \tag{12.66}$$

Let

$$V = H_{0,\Gamma_N}(\text{div }, \Omega) = \{v \in H(\text{div }, \Omega) : v \cdot n = 0 \text{ on } \Gamma_N\}, \quad Q = L^2(\Omega)$$

we have the following variational form of mixed type:

$$\begin{cases} a(u, v) + b(v, p) = 0 & \forall\, v \in V, \\ b(u, q) = -(f, q) & \forall\, q \in Q, \end{cases} \tag{12.67}$$

where

$$a(u, v) = (\alpha^{-1}u, v), b(v, q) = (\text{div } v, q).$$

One of the key point for the mixed formulation is that the natural boundary condition of p transforms into the essential boundary condition of u, and the essential boundary of p transforms into the natural boundary of p.

We want to note that if $\Gamma_D = \emptyset$, a compatible condition

$$\int_{\partial\Omega} \alpha p_n + \int_\Omega f = 0$$

is needed. And we have to impose a condition to p (e.g. $\int_\Omega p = 0$) to guarantee the uniqueness.

Lemma 93. *The variational problem* (12.67) *is well-posed.*

Proof. By Brezzi's theorem, we only need to verify two inf-sup conditions. The first inf-sup condition concerning the bilinear for $a(\cdot, \cdot)$ is easy, since for $p \in V$ such that div $v = 0$, we have

$$a(v, v) = (\alpha^{-1}v, v) \geq \alpha_1^{-1}\|v\|^2 = \alpha_1^{-1}\|v\|^2_{H(div)}.$$

For the second inf-sup condition concerning $b(\cdot, \cdot)$, for any $q \in Q$, consider the auxiliary problem

$$\begin{aligned} -\Delta\phi &= q \quad \text{in } \Omega \\ \phi &= 0 \quad \text{on } \Gamma \end{aligned}$$

Take $u = \nabla\phi \in Q$, it is then easy to see that

$$(\mathrm{div} u, q) \geq c_0\|q\|^2.$$

Since $\|u\|_V^2 = \|u\|_0^2 + \|\mathrm{div} u\|_0^2 = \|\nabla\phi\|_0^2 + \|\Delta\phi\|_0^2 \leq C\|q\|_0^2$,

$$\sup_{v\in V}\frac{b(v,q)}{\|v\|_V} \geq \beta\|q\|_Q.$$

This gives rise to the second inf-sup condition. □

More subtleties get involved when we consider choice of finite element pair for velocity and pressure. For example, if we consider piecewise linear function for velocity and piecewise constant for pressure, the problem cannot be well-posed on the mesh below since there will be more number of degrees of freedom for pressure than for velocity. Other choices like $(P_2^0 - P_0^{-1})$ pair may resolve this issue.

12.9.1 Raviart-Thomas mixed finite element approximations

Consider the finite element spaces

$$V_h = H_h^{\mathrm{div}}(\Omega) \cap H_{0,\Gamma_N}(\mathrm{div}\ ,\Omega), \quad Q_h = L_h^2(\Omega)$$

Find $u_h \in V_h, p_h \in Q_h$ such that

$$\begin{cases} (u_h, v_h) + (\mathrm{div}\ v_h, p_h) = 0 & \forall\ v_h \in V_h, \\ \qquad (\mathrm{div}\ u_h, q_h) = -(f, q_h) & \forall\ q_h \in Q_h, \end{cases} \tag{12.68}$$

By Brezzi theory, for the well-possedness and the approximation property of this discretized variational problem, we need to verify two inf-sup condititons. Again the first inf-sup condition is trivial. Assume that Ω is convex. To see the second inf-sup condition, for any $q_h \in Q_h$, consider the auxiliary problem

$$\begin{aligned} -\Delta\phi &= q_h \quad \text{In } \Omega \\ \phi &= 0 \quad \text{on } \partial\Gamma_D \\ \frac{\partial}{\partial n}\phi &= 0 \quad \text{on } \partial\Gamma_N \end{aligned}$$

We have $\|\phi\|_{2,\Omega} \lesssim \|q_h\|_{0,\Omega}$. Take $u_h = -\Pi_h^{\mathrm{div}}\nabla\phi \in V_h$, it is then easy to see that

$$(\mathrm{div} u_h, q_h) = (\Pi_h^0 q_h, q_h) \geq c_0\|q_h\|_Q^2.$$

Since $\mathrm{div} u_h \in Q_h$,

$$\|\mathrm{div} u_h\|_0^2 = (-\mathrm{div}\Pi_h^{\mathrm{div}}\nabla\phi, \mathrm{div} u_h) = (q_h, \mathrm{div} u_h),$$
$$\|\mathrm{div} u_h\|_0 \lesssim \|q_h\|_Q.$$

Together with $\|u_h\|_0 = \|\Pi_h^{\mathrm{div}}\nabla\phi\|_0 \lesssim \|\nabla\phi\|_1 \lesssim \|q_h\|_Q$, we have

$$\|u_h\|_V \leq C\|q_h\|_Q.$$

Thus,

$$\sup_{v_h\in V_h}\frac{b(v_h,q_h)}{\|v_h\|_V} \geq \beta\|q\|_Q,$$

and gives the 2nd inf-sup condition.

12.9.2 Error estimates

By our basic theory, we have the following estimate

$$\|u - u_h\|_{\mathrm{H(div)}} + \|p - p_h\|_0 \lesssim \inf_{v_h \in V_h, q_h \in Q_h} (\|u - v_h\|_{\mathrm{H(div)}} + \|p - q_h\|_0)$$
$$\leq \|u - \Pi_h^{\mathrm{div}} u\|_{\mathrm{H(div)}} + \|p - \Pi_h^0 p\|_0.$$

We have proved before that

$$\|u - \Pi_h^{\mathrm{div}} u\|_0 \lesssim h^m |u|_m, \quad 1 \leq m \leq k + 1.$$

Furthermore

$$\|\mathrm{div}\ (u - \Pi_h^{\mathrm{div}} u)\|_0 \lesssim \|\mathrm{div}\ u - \Pi_h^0 \mathrm{div}\ u)\|_0 \lesssim h^m |\mathrm{div}\ u|_m, \quad 1 \leq m \leq k.$$

Consequently, we have

$$\|u - u_h\|_{\mathrm{H(div)}} + \|p - p_h\|_0 \lesssim h^m (|u|_m + |\mathrm{div}\ u|_m + |p|_m) \quad 1 \leq m \leq k + 1.$$

In the rest of this section, we shall give some more refined analysis under the assumption that Ω has smooth boundary or it is convex Liptschitz domain. For simplicity, we assume that $\Gamma_N = \emptyset$.

We assume that $(u, p) \in V \times Q$ and $(u_h, p_h) \in V_h \times Q_h$ are solutions to (12.67) and (12.68) respectively.

Lemma 94. *Then*

(12.69) $$\|u - u_h\| \lesssim h\|f\|.$$

and

(12.70) $$\mathrm{div}\ u_h = \Pi_h^0 f$$

and, if $(u_H, p_H) \in V_H \times Q_H \subset V_h \times Q_h$ *is a mixed approximation to* (12.67) *on a coarser mesh with* $H \geq h$, *then*

(12.71) $$\|u_h - u_H\| \lesssim H\|\mathrm{div}\ u_h\|$$

and

(12.72) $$\|\mathrm{div}\ (u_h - u_H)\| \lesssim H\|\mathrm{grad}_h \Pi_h^{\mathrm{div}} f\| = H\|\mathrm{grad}_h \mathrm{div}\ _h u_h\|$$

and

(12.73) $$\|u_h - u_H\|_{\mathrm{div}} \lesssim H\|\mathbf{A}_h^{\mathrm{div}} u_h\|.$$

Proof. The first estimate is easy

$$\|u - u_h\| \leq \|u - \Pi_h^{\mathrm{div}} u\| \lesssim h\|u\|_1 \lesssim h\|p\|_2 \lesssim h\|f\|$$

where we have used the H^2 regularity theory the original boundary value problem on smooth or convex Liptschiz domain.

Let $f_h = \Pi_h^0 f$ and $w \in H^1(\Omega)^3$ be such that

$$\mathrm{div}\ w = f_h - \Pi_H^0 f_h, \quad \|w\|_1 \lesssim \|f_h - \Pi_H^0 f_h\|.$$

It follows that

$$\|f_h - \Pi_H^0 f_h\|^2 = (\mathrm{div}\ w, f_h - \Pi_H^0 f_h) = ((\Pi_h^0 - \Pi_H^0)\mathrm{div}\ w, f_h)$$
$$= (\mathrm{div}\ (\Pi_h^{\mathrm{div}} - \Pi_H^{\mathrm{div}})w, f_h) = -((\Pi_h^{\mathrm{div}} - \Pi_H^{\mathrm{div}})w, \mathrm{grad}_h f_h)$$
$$\lesssim H\|w\|_1\|\mathrm{grad}_h f_h\| \lesssim H\|f_h - \Pi_H^0 f_h\|\|\mathrm{grad}_h f_h\|.$$

This proves that $\|f_h - \Pi_H^0 f_h\| \lesssim H\|\mathrm{grad}_h f_h\|$. With $\mathrm{div}\ (u - u_H) = f_h - \Pi_H^0 f_h$, the desired result then follows.
□

12.10 A model problem involving $H(\text{curl})$ and $H(\text{div})$ spaces

We shall now study a model problem that is useful for the study of the curl operator and its relevant finite element discretization. First we introduce the following spaces:

$$Z(\text{div}) = \{v \in H(\text{div}) : \text{div } v = 0\} \tag{12.74}$$

and

$$U = Z_0(\text{div}) \cap H_0(\text{curl}). \tag{12.75}$$

The strong form of our model problem is: Given $f \in Z_0(\text{div})$, find $p \in U$ such that

$$\begin{aligned} \text{curl curl } p &= f \quad \text{in } \Omega,\\ p \times n &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{12.76}$$

The primal variational formulation is: Find $p \in U$ such that

$$(\text{curl } p, \text{curl } q) = (f, q) \quad \forall\, q \in U. \tag{12.77}$$

This problem is well-posed as $Z_0(\text{div})$ can be proven to be a Hilbert space under the norm $\|\text{curl } p\|$. (It is easy to see that $\|\text{curl } p\|$ is indeed a norm: if curl $p = 0$, we have $p = \text{grad}\phi$ for some $\phi \in H^1(\Omega)$. Since $p \in Z_0(\text{div})$, $\Delta\phi = 0$ and grad $\phi \times n = 0$ on $\partial\Omega$ means that ϕ is constant everywhere in Ω. Thus $p = \text{grad } \phi = 0$.

We now introduce a new variable:

$$u = \text{curl } p \in \text{curl} \tag{12.78}$$

and define a mixed formation for (12.76) as follows:

$$\begin{aligned} -\text{div}(\alpha(x)\text{grad } p) &= f \quad \text{in } \Omega,\\ p &= 0 \quad \text{on } \Gamma_D\\ \frac{\partial p}{\partial n} &= 0 \quad \text{on } \Gamma_N.\\ (u, v) - (\text{curl } v, p) &= 0 \quad v \in \text{curl},\\ (\text{curl } u, q) &= (f, q) \quad q \in Z(\text{div}). \end{aligned} \tag{12.79}$$

Lemma 95. *If Ω has smooth boundary or it is convex Lipschitz, then $\in H^1$ and $u = \text{curl } p \in H^1$ and*

$$\|p\|_1 \lesssim \|u\| \text{ and } \|u\|_1 \lesssim \|f\|. \tag{12.80}$$

The finite element approximation to (12.76) is: Find $(u_h, p_h) \in Z_h(\text{div}) \times H_h(\text{div})$ such that

$$\begin{aligned} (u_h, v_h) - (\text{curl } v_h, p_h) &= 0 \quad v_h \in H_h(\text{curl}),\\ (\text{curl } u_h, q_h) &= (f, q_h) \quad q \in Z_h(\text{div}). \end{aligned} \tag{12.81}$$

We have the error equation:

$$\begin{aligned} (u - u_h, v_h) - (\text{curl } v_h, p - p_h) &= 0 \quad v_h \in H_h(\text{curl}),\\ (\text{curl } (u - u_h), q_h) &= 0 \quad q \in Z_h(\text{div}). \end{aligned} \tag{12.82}$$

It is easy to see that the solution to the above discrete problem is given by

$$u_h = \text{grad}_h p_h, \quad \text{curl } p_h = Q_{Z_h} f. \tag{12.83}$$

where $Q_{Z_h} : L^2 \mapsto Z_h(\mathrm{div})$ is the L^2 projection.

One complication in the analysis of this mixed method is, in general

$$\mathrm{curl}\, \Pi_h^{\mathrm{curl}} u \neq Q_{Z_h} \mathrm{curl}\, u \text{ although } \mathrm{curl}\, \Pi_h^{\mathrm{curl}} u = \Pi_h^{\mathrm{div}} \mathrm{curl}\, u.$$

But if we assume that $f = \mathrm{curl}\, u \in H_h(\mathrm{curl})$ (which implies $f \in Z_h(\mathrm{div})$), then

$$\mathrm{curl}\, u_h = Q_{Z_h} f = f = \mathrm{curl}\, u = \Pi_h^{\mathrm{div}} \mathrm{curl}\, u = \mathrm{curl}\, \Pi_h^{\mathrm{curl}} u.$$

Thus $\Pi_h^{\mathrm{curl}} u - u_h$ is curl free and taking $v_h = \Pi_h^{\mathrm{curl}} u - u_h$ in the error equation 12.82 we obtain

$$(u - u_h, \mathrm{curl}\, \Pi_h^{\mathrm{curl}} u - u_h) = 0.$$

This leads to

$$\|u - u_h\| \leq \|u - \Pi_h^{\mathrm{curl}} u\| \lesssim h|u|_1, \quad u \in H^1 \text{ such that } \mathrm{curl}\, u \in H_h(\mathrm{div}). \tag{12.84}$$