3

Finite Difference Method

In this chapter, we will consider finite difference method for solving the following model problem on $\Omega \subset \mathbb{R}^d$ $(d = 1, 2)$:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

The idea is to use this example in order to determine how the linear algebraic systems are related to partial differential equations (PDEs). Then we consider the error estimates for the finite difference schemes by using maximum principle.

3.1 Finite difference method

Finite difference method is a more straightforward discretization scheme than the finite element method. Let us see how it works. We will consider the case $d = 1, 2$. The generalization to $d = 3$ is straightforward.

$d = 1$

For $d = 1$, we have $\Omega = (0, 1)$, which is the unit interval. We consider a uniform grid, denoted by $\mathcal{T}_h$, of the interval $[0, 1]$ as follows:

$$0 = x_0 < x_1 < \cdots < x_{N+1} = 1, \quad x_j = \frac{j}{N+1}, \quad (j = 0, \cdots, N+1), \tag{3.2}$$

where N is a given integer. This partition consists of uniform subintervals with the length $h = \frac{1}{N+1}$. Such a uniform partition is shown in Figure 3.1.

A derivative is a limit of some difference quotient. Therefore, in accord with the basic idea of the difference method, we use the difference quotient to approximate the derivatives. For the model problem 23.7 in one dimension, we can use the center difference to approximate the second derivative in the Laplace operator as follows:

$$u''(x_j) \approx \frac{u'(x_{j+1/2}) - u'(x_{j-1/2})}{h},$$

where $x_{j+1/2} = (x_{j+1} + x_j)/2$. Similarly, the two first-order derivatives can also be approximated by the center difference:

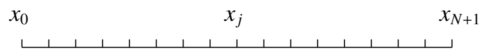

Fig. 3.1. One-dimensional uniform grid

$$u'(x_{j+1/2}) \approx \frac{u(x_{j+1}) - u(x_j)}{h},$$
$$u'(x_{j-1/2}) \approx \frac{u(x_j) - u(x_{j-1})}{h}.$$

Thus, for a smooth solution u, we have

$$u''(x_j) = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} + O(h^2). \tag{3.3}$$

The error term in the above (3.3) is often called the truncation error or consistency error. Actually, by Taylor's expansion, we have

$$u(x_{j-1}) = u(x_j - h) = u(x_j) - u'(x_j)h + \frac{1}{2}u''(x_j)h^2 - \frac{1}{3!}u^{(3)}(x_j)h^3 + \frac{1}{4!}u^{(4)}(\hat{x}_j^-)h^4,$$
$$u(x_{j+1}) = u(x_j + h) = u(x_j) + u'(x_j)h + \frac{1}{2}u''(x_j)h^2 + \frac{1}{3!}u^{(3)}(x_j)h^3 + \frac{1}{4!}u^{(4)}(\hat{x}_j^+)h^4,$$

so we add the above two equations together to get

$$u(x_{j-1}) + u(x_{j+1}) = 2u(x_j) + u''(x_j)h^2 + \frac{2}{4!}(u^{(4)}(\hat{x}_j^-) + u^{(4)}(\hat{x}_j^+))h^4,$$

i.e.,

$$u''(x_j) = \frac{u(x_{j-1}) - 2u(x_j) + u(x_{j+1})}{h^2} - \frac{2}{4!}(u^{(4)}(\hat{x}_j^-) + u^{(4)}(\hat{x}_j^+))h^2.$$

Therefore, we can write our first finite difference scheme as follow. If $\mu = (\mu_i) \in \mathbb{R}^N$ satisfies

$$\frac{-\mu_{j-1} + 2\mu_j - \mu_{j+1}}{h^2} = f_j, \quad 1 \le j \le N, \quad \mu_0 = \mu_{N+1} = 0, \tag{3.4}$$

where

$$f_j = f(x_j). \tag{3.5}$$

Thanks to (3.3), as $h \to 0$, we expect that

$$\mu_j \approx u(x_j).$$

It is natural to expect this approximation to become more accurate as N gets larger, or in other words, as h becomes smaller. In fact, it can be proved that, if u is smooth,

$$\mu_j - u(x_j) = O(h^2). \tag{3.6}$$

Naturally, there is a close correlation of this error with the consistency error term in (3.3). We shall discuss about their relationship.

Linear algebraic system

The equation (3.4) can be written as the following linear system of equations:

$$A\mu = b \tag{3.7}$$

where $A \in \mathbb{R}^{N\times N}$,

$$A = \text{tridiag}(-1, 2, -1), \tag{3.8}$$

and $b = (h^2 f(x_i))$.

$d = 2$

Next, we consider the two-dimensional case in which $\Omega = [0,1] \times [0,1]$. As with the one-dimensional case, we consider a uniform triangulation of Ω. For the x direction, we use the same partition 3.2. For the y direction, we consider a similar partition:

$$0 = y_0 < y_1 < \cdots < y_{N+1} = 1, \quad y_j = \frac{j}{N+1}, \quad (j = 0, \cdots, N+1). \tag{3.9}$$

Such a uniform partition in the x and y directions leads us to a special example in two dimensions, a uniform square mesh $\mathbb{R}_h^2 = \{(nh, mh); n, m \in \mathbb{Z}\}$ (Figure 6). Let $\Omega_h = \Omega \cap \mathbb{R}_h^2$, the set of interior mesh points and $\partial\Omega_h = \partial\Omega \cap \mathbb{R}_h^2$, the set of boundary mesh points.

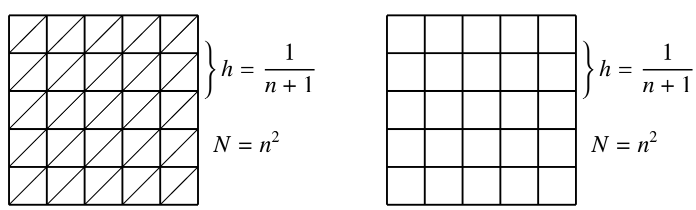

Fig. 3.2. Two-dimensional uniform grid for finite element and finite difference

As with the one-dimensional case, we can use the center difference to approximate the derivatives as follows:

$$\frac{\partial^2 u}{\partial x^2}(x_i, y_j) = \frac{u(x_{i+1}, y_j) - 2u(x_i, y_j) + u(x_{i-1}, y_j)}{h^2} + O(h^2),$$
$$\frac{\partial^2 u}{\partial y^2}(x_i, x_j) = \frac{u(x_i, y_{j+1}) - 2u(x_i, y_j) + u(x_i, y_{j-1})}{h^2} + O(h^2).$$

Thus,

$$(-\Delta_h u)(x_i, y_j) = (-\Delta u)(x_i, y_j) + O(h^2) \tag{3.10}$$

where $-\Delta_h$ is a discretized operator for $-\Delta$ given by

$$(-\Delta_h u)(x_i, y_j) = \frac{1}{h^2}(4u(x_i, y_j) - (u(x_{i+1}, y_j) + u(x_{i-1}, y_j) + u(x_i, y_{j+1}) + u(x_i, y_{j-1}))). \tag{3.11}$$

The truncation error (3.10) can be proved by applying Taylor's expansion directly, which is similar how we did for 1-d case.

The finite difference scheme is then formed by

$$4\mu_{ij} - (\mu_{i+1,j} + \mu_{i-1,j} + \mu_{i,j+1} + \mu_{i,j-1}) = h^2 f_{i,j}, \tag{3.12}$$

where

$$f_{i,j} = f(x_i, y_j). \tag{3.13}$$

By putting the two-dimensional nodal points (x_i, y_j) in lexicographical ordering as

$$x^h_{(j-1)n+i} = (x_i, y_j), \quad 1 \le i, j \le n,$$

we can write (3.12) as

$$A\mu = b, \tag{3.14}$$

with

$$A = tridiag(-I, B, -I), \quad B = tridiag(-1, 4, -1). \tag{3.15}$$

The matrix A is a block tridiagonal matrix.

3.2 Maximum principle and error analysis

To give an error estimate of the five-point stencil (3.12), we give the following maximum principle [**?**].

Maximum principle

- If $Lu \ge 0$ (or (≤ 0) inside Ω then u achieve its minimum (or maximum) on the boundary $\partial\Omega$.
- If $Lu \ge 0$ and $u|_{\partial\Omega} \ge 0$, then $u \ge 0$.

We also have discrete maximum principle.

Theorem 22. *If $-\Delta_h u \le 0$ on Ω_h, then $\max_{\Omega_h} u \le \max_{\partial\Omega_h} u$. Furthermore, $\max_{\Omega_h} u = \max_{\partial\Omega_h} u$ if and only if u is a constant.*

Proof. Suppose $\max_{\Omega_h} u \ge \max_{\partial\Omega_h} u$, take $x_0 \in \Omega_h$ such that $u(x_0) = \max_{\Omega_h} u$, and x_i with $i = 1, 2, 3, 4$ as its four nearest neighbors. Then we have

$$4u(x_0) \le \sum_{i=1}^{4} u(x_i) - h^2 \Delta_h u(x_0) \le \sum_{i=1}^{4} u(x_i) \le 4u(x_0),$$

so $u(x_i) = u(x_0)$ also achieve the maximum value as well. We can apply the same argument thoughout all points on Ω_h, and finally we conclude that u is a constant. □

Remark 4. (1) Note that the analogous discrete minimum principle can be given by replacing max by min and reversing the inequalities.

(2) One application of discrete maximum principle is to prove the uniqueness of the problem (3.12). Actually, we only need to prove that if $\Delta_h u_h = 0$ and $u_h|_{\partial\Omega_h} = 0$, and then $u_h = 0$ on Ω_h. By discrete maximum and minimum principle, we can see that $u_h = 0$ everywhere.

The next application of maximum principle is proof of maximum norm stability.

Theorem 23. *The solution u_h of problem* (3.12) *satisfies*

$$\|u_h\|_{L^\infty(\bar{\Omega}_h)} \le \frac{1}{8}\|f\|_{L^\infty(\Omega_h)}. \tag{3.16}$$

Proof. Introduce the comparison function $g(x,y) = [(x-1/2)^2 + (y-1/2)^2]/4$ satisfying $\Delta_h g = 1$ on Ω_h, and $0 \le g \le 1/8$ on $\bar{\Omega}_h$. Let $M = \|f\|_{L^\infty(\Omega_h)}$. Then

$$\Delta_h(u_h + Mg) = \Delta_h u_h + M \ge 0,$$

so

$$\max_{\Omega_h} u_h \le \max_{\Omega_h}(u_h + Mg) \le \max_{\partial\Omega_h}(u_h + Mg) \le \max_{\partial\Omega_h} u_h + \frac{1}{8}M = \frac{1}{8}M.$$

Applying the similar argument to $-u_h$, we can prove the theorem. □

Since $-\Delta_h u_h = f = -\Delta u$ on Ω_h, applying the above theorem, we can obtain the following inequality

$$\|u - u_h\|_{L^\infty(\Omega_h)} \le \frac{1}{8}\|\Delta_h u - \Delta_h u_h\|_{L^\infty(\Omega_h)}. \tag{3.17}$$

Furthermore, if $u \in C^4(\bar{\Omega})$, then combining (3.10) and (3.17), we obtain

$$\|u - u_h\|_{L^\infty(\Omega_h)} \le Ch^2.$$

3.3 Finite difference methods on curved boundaries.

In this section, we consider finite difference methods for Poisson's problem (23.7) for $d = 2$ on a more general domain [**?**]. Let $\Omega \subset \mathbb{R}^2$ be a bounded open set with smooth boundary $\partial\Omega$. Set $\Omega_h = \Omega \cap \mathbb{R}_h^2$, the interior mesh points. Each interior mesh point needs to have four neighbor points. We denote Ω_h^0 as the set of points in Ω_h all four of whose neighbor belong to Ω_h. For any $(x,y) \in \Omega_h^\partial = \Omega_h \backslash \Omega_h^0$, we find other points on $\partial\Omega$ such that $(x+sh, y+th) \in \partial\Omega$ with $-1 \le s, t \le 1$.

First, we consider to use the linear combination of $u(-h_n)$, $u(0)$, and $u(h_p)$ to approximate $\frac{d^2u}{dx^2}(0)$. By Taylor's expansion, we have

$$\begin{aligned} a_1 u(-h_n) + a_2 u(0) + a_3 u(h_p) =& (a_1 + a_2 + a_3)u(0) + (a_3 h_p - a_1 h_n)u'(0) \\ &+ \frac{1}{2}(a_3 h_p^2 + a_1 h_n^2)u''(0) + \frac{1}{6}(a_3 h_p^3 - a_1 h_n^3)u'''(0) + \cdots . \end{aligned}$$

To obtain a consistent approximation we set

$$a_1 + a_2 + a_3 = 0, \quad a_3 h_p - a_1 h_n = 0, \quad a_3 h_p^2 + a_1 h_n^2 = 1,$$

which give

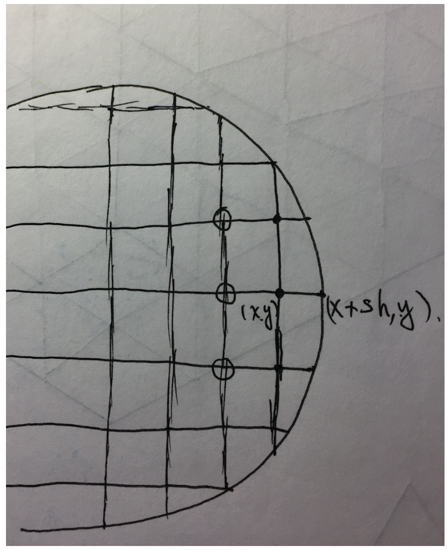

Fig. 3.3. Finite differenece method on curved domain

$$a_1 = \frac{2}{h_n(h_n + h_p)}, \quad a_2 = -\frac{2}{h_n h_p}, \quad a_3 = \frac{2}{h_p(h_n + h_p)}.$$

For 2-d case, let $(x - h_W, y)$, $(x + h_E, y)$, $(x, y - h_S)$ and $(x, y + h_N)$ be the four neighbor points. We apply the same consideration to both $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ to get the Shortley-Weller scheme for $\Delta_h u$,

$$\Delta_h u(x, y) = f(x, y), \tag{3.18}$$

where

$$\begin{aligned}\Delta_h u(x, y) =& \frac{2}{h_E(h_E + h_W)} u_E + \frac{2}{h_W(h_E + h_W)} u_W \\ &+ \frac{2}{h_N(h_N + h_S)} u_N + \frac{2}{h_S(h_N + h_S)} u_S - \left(\frac{2}{h_E h_W} + \frac{2}{h_N h_S}\right) u_0\end{aligned}$$

For $u \in C^3(\Omega)$, using Taylor's expansion, we can obtain by direct calculation,

$$\|\Delta u - \Delta_h u\|_{L^\infty(\Omega_h)} \leq \frac{2M_3}{3} h, \tag{3.19}$$

where M_3 is the maximum of the L^∞ norms of the third derivatives of u. Hence, the truncation error for the mesh points nearing the boundary is reduced to $O(h)$, but at the interior mesh points in Ω_h^0, the truncation error is still bounded by $O(h^2)$ as before.

Generally, the matrix derived from Shortley-Weller scheme (3.18) is not symmetric, but it maintains other good properties in the case of the square: (1) it is sparse, with at most five elements per row; (2) it has nonnegative diagonal elements and negative off-diagonal elements; (3) it is diagonally dominant. By these properties, we can obtain the discrete maximum principle with the similar proof as we did for Theorem 22, and a stability result with the similar proof in Theorem 23. Hence, we can easily obtain an $O(h)$ convergence result. In the following, we will show that, with a more carefully analysis, even though the truncation error is only $O(h)$ at points belonging to Ω_h^∂, the error is $O(h^2)$ at all mesh points.

Let V_h denote the space of mesh functions defined on $\Omega_h^0 \cup \Omega_h^\partial$, and which vanish on Ω_h^∂. We use maximum norm for V_h. Let W_h denote the space of mesh functions defined on Ω_h^0, and we use a different norm for it, i.e.,

$$\|v_h\|_{W_h} = \max\left\{\max_{x\in\Omega_h^0}|v_h(x)|, h\max_{x\in\Omega_h^\partial}|v_h(x)|\right\}.$$

This norm is smaller than the maximum norm, but in this norm, the truncation error is

$$\|\Delta u - \Delta_h u\|_{W_h} \le O(h^2),$$

because the weight h decreases the influence of error on the points with a neighbor on the boundary.

Next, we will show that the Shortley-Weller discrete Laplacian is stable from V_h to W_h with this new norm. The key point of proof is to construct a new comparison function. Suppose that Ω is contained in the disk with radius r at center point $p = (p_1, p_2)$. Then we define comparison function

$$g(x,y) = \begin{cases} \frac{(x-p_1)^2+(y-p_2)^2}{4}, & (x,y)\in\Omega_h, \\ \frac{(x-p_1)^2+(y-p_2)^2}{4} + h, & (x,y)\in\partial\Omega_h. \end{cases}$$

Note that $\|g(x,y)\|_{L^\infty} \le r^2/4 + h \le r^2/4 + 2r$. Furthermore, $\Delta_h g(x,y) = 1$ if $x \in \Omega_h^0$ because it is quadratic at (x,y) and all its neighbors. But if $(x,y)\in\Omega_h^\partial$, then there is an additional term in $\Delta_h g(x,y)$ for each neighbor of (x,y) on the boundary. For example, if $(x-h_w, y)\in\partial\Omega_h$ is a neighbor of (x,y) and the other neighbors are in Ω_h^0, then

$$\Delta_h g(x,y) = 1 + \frac{2h}{h_w(h_w+h)} \ge h^{-1},$$

due to $h_w \le h$. Thus we have

$$\Delta_h g(x,y) \ge \begin{cases} 1, & (x,y)\in\Omega_h^0, \\ h^{-1}, & (x,y)\in\Omega_h^\partial. \end{cases}$$

Let $M = \|\Delta_h v_h\|_{W_h}$ where v_h is a mesh function defined on $\bar{\Omega}_h = \Omega_h \cup \partial\Omega_h$. If $(x,y)\in\Omega_h^0$, then $M \ge |\Delta_h v_h(x,y)|$ and $\Delta_h g(x,y) = 1$, so

$$\Delta_h(Mg(x,y)) \ge |\Delta_h v_h(x,y)|.$$

If $(x,y)\in\Omega_h^\partial$, then $M \ge h|\Delta_h v_h(x,y)|$ and $\Delta_h g(x,y) \ge h^{-1}$, so

$$\Delta_h(Mg(x,y)) \ge |\Delta_h v_h(x,y)|$$

as well. Therefore, we have

$$\Delta_h(v_h + Mg) \ge 0 \quad \text{on } \Omega_h.$$

We can then apply the maximum principle to get

$$\max_{\bar{\Omega}_h} v_h \le \max_{\bar{\Omega}_h}(v_h + Mg) \le \max_{\partial\Omega_h}(v_h + Mg) \le \max_{\partial\Omega_h} v_h + \|g\|_{L^\infty}\|\Delta_h v_h\|_{W_h}.$$

Let $C = \|g\|_{L^\infty}$, and applying the same argument for $-v_h$, we have

$$\|v_h\|_{L^\infty(\bar{\Omega}_h)} \le \|v_h\|_{L^\infty(\partial\Omega_h)} + C\|\Delta_h v_h\|_{W_h}.$$

If $v_h \in V_h$ vanishes on $\partial\Omega_h$, then

$$\|v_h\|_{L^\infty(\bar{\Omega}_h)} \leq C\|\Delta_h v_h\|_{W_h}.$$

Therefore, we apply the above stability estimate to $v_h = u - u_h$, and obtain the error estimate

$$\|u - u_h\|_{L^\infty(\bar{\Omega}_h)} \leq C\|\Delta_h u - \Delta_h u_h\|_{W_h} = O(h^2).$$