15

Discontinuous Galerkin Methods

In this chapter, we will introduce discontinuous Galerkin (DG) Methods.

15.1 Introduction

The initial DG method was introduced by Reed and Hill (1973) for numerically solving the neutron transport equation.

$$\sigma u + \nabla \cdot (\mathbf{a}u) = f \quad \text{in}\, \Omega,$$

where σ is a real number and $\mathbf{a}$ a constant vector.

In recent years, due to the flexibility in constructing feasible local shape function spaces and the advantage to capture non-smooth or oscillatory solutions effectively, DG methods have been applied to a wide range of partial differential equations, such as convection-diffusion equations, Navier-Stokes equations, Hamilton-Jacobi equations, the radiative transfer equation and so on.

DG methods differ from the standard finite element methods in that functions are allowed to be discontinuous across the element boundaries. Since no interelement continuity is required, DG methods allow general meshes with hanging nodes and elements of different shapes. The advantages of this include the ease of using polynomial functions of different orders in different elements (p-adaptivity), more flexibility in mesh refinements (h-adaptivity), and the locality of the discretization, which makes them ideally suited for parallel computing. Their compact formulation can be applied near boundaries without special treatment, which greatly increases the robustness and accuracy of any boundary condition implementation. Furthermore, DG methods is locking-free for the elasticity problem, and continuous DG (CDG) method handles biharmonic problem very well.

DG methods for elliptic equations were independently proposed in the 1970s. Many variants were introduced and studied, which were generally called interior penalty (IP) methods. Their development was independent of that of the DG methods for hyperbolic equations. There are two basic ways to construct DG methods for elliptic problems. The first way is to add a penalty term into the bilinear form, penalizing the interelement discontinuity. The second one is to choose suitable numerical fluxes to make the DG schemes consistent, conservative, and stable. In particular, it can be shown that the methods of the first family, those based on the choice of the bilinear form, can be obtained as special cases of the second family simply by choosing proper numerical fluxes.

In this chapter, we will take Poisson equation again as a model problem to construct DG methods on $\Omega = (0,1)^d$ $(d = 1, 2)$:

$$\begin{cases} -\Delta u = f, & \text{in}\ \Omega, \\ \quad u = g, & \text{on}\ \partial\Omega, \end{cases} \tag{15.1}$$

15.1.1 Notation

We introduce some notations before giving DG formulations. Given a bounded domain $D \in \mathbb{R}^2$ and a positive integer m, $H^m(D)$ is the Sobolev space with the corresponding usual norm and semi-norm, which are denoted respectively by $\|\cdot\|_{m,D}$ and $|\cdot|_{m,D}$. We abbreviate them by $\|\cdot\|_m$ and $|\cdot|_m$ respectively when D is chosen as Ω. $\|\cdot\|_D$ is the norm of Lebesgue space $L^2(D)$. We assume Ω is a polygonal domain and denote by $\{\mathcal{T}_h\}_h$ a family of triangulations of $\overline{\Omega}$, with the minimal angle condition satisfied. Let $h_K = \mathrm{diam}(K)$ and $h = \max\{h_K : K \in \mathcal{T}_h\}$. $H^1(\mathcal{T}_h) = \{v \in L^2(\Omega), v|_K \in H^1(K)\ \forall K \in \mathcal{T}_h\}$. Denote by $\mathcal{E}_h$ the union of the boundaries of the elements K of $\mathcal{T}_h$, $\mathcal{E}_h^i = \mathcal{E}_h \backslash \partial\Omega$ is the set of interior edges and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ is the set of boundary edges. The traces of functions in $H^1(\mathcal{T}_h)$ belong to $T(\mathcal{E}_h) := \Pi_{K\in\mathcal{T}_h} L^2(\partial K)$. Note that $v \in T(\mathcal{E}_h)$ is double-valued on $\mathcal{E}_h^i$ and single-valued on $\partial\Omega$. $L^2(\mathcal{E}_h)$ can be regarded as the subspace of $T(\mathcal{E}_h)$ consisting of functions whose two values coincide on all internal edges.

Let $p \geq 1$ be the degree of polynomial function and introduce the following finite element spaces:

$$\begin{aligned} V_h &= \{v_h \in L^2(\Omega) : \ v_h|_K \in P_p(K)\ \forall\, K \in \mathcal{T}_h\}, \\ W_h &= \{\mathbf{w}_h \in [L^2(\Omega)]^2 : \ \mathbf{w}_h|_K \in [P_p(K)]^2\ \forall\, K \in \mathcal{T}_h\}. \end{aligned}$$

For $v \in H^1(\mathcal{T}_h)$, ∇_h is the broken gradient which is defined by the relation $\nabla_h v = \nabla v$ on any element $K \in \mathcal{T}_h$.

Let e be the common edge of two elements K^+ and K^-, and $n^i = n|_{\partial K^i}$ be the unit outward normal vector on ∂K^i with $i = +, -$. For $v \in T(\mathcal{E}_h)$, let $v^i = v|_{\partial K^i}$, and similarly, for $q \in [T(\mathcal{E}_h)]^2$, we denote $\mathbf{q}^i = \mathbf{q}|_{\partial K^i}$. Then we define the average $\{\cdot\}$ and the jump $[\cdot]$ on $e \in \mathcal{E}_h^i$ by

$$\begin{aligned} \{v\} &= \frac{1}{2}(v^+ + v^-), & [v] &= v^+ n^+ + v^- n^-, \\ \{\mathbf{q}\} &= \frac{1}{2}(\mathbf{q}^+ + \mathbf{q}^-), & [\mathbf{q}] &= \mathbf{q}^+ \cdot n^+ + \mathbf{q}^- \cdot n^-. \end{aligned}$$

If $e \in \mathcal{E}_h^\partial$, we set

$$[v] = vn, \quad \{\mathbf{q}\} = \mathbf{q} \quad \text{on } e \in \mathcal{E}_h^\partial,$$

where n is the outward unit normal.

Let us give the following identities which are used often in this chapter. For any scalar-valued function v and any vector-valued function $\mathbf{w}$, both being continuously differentiable over K, we have the following integration by parts formula:

$$\int_K \nabla v \cdot \mathbf{w}\, dx = -\int_K v \nabla \cdot \mathbf{w}\, dx + \int_{\partial K} v\mathbf{n} \cdot \mathbf{w} ds. \tag{15.2}$$

For a scalar-valued function v and a vector-valued function $\mathbf{w}$, after a direct manipulation, we have

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} (v\mathbf{n}_K) \cdot \mathbf{w}\, ds = \sum_{e\in\mathcal{E}_h} \int_e [v] \cdot \{\mathbf{w}\}\, ds + \sum_{e\in\mathcal{E}_h^i} \int_e \{v\}[\mathbf{w}]\, ds. \tag{15.3}$$

15.2 (Non-symmetric) Interior penalty (IP) method

Consider an elliptic problems with Dirichlet boundary condition, for example,

$$-\Delta u = f \text{ in } \Omega \quad \text{and} \quad u = g \text{ on } \partial\Omega,$$

where $f \in L^2(\Omega)$ and $g \in H^{-1/2}(\partial\Omega)$. Dirichlet boundary condition is not included naturally as Neumann boundary condition in the week formulation. If we replace the Dirichlet boundary condition with the approximate boundary condition $u_\mu + \mu^{-1}\partial u_\mu/\partial n = g$ (μ is a large positive parameter), we can get the following weak form: Find $u_\mu \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u_\mu \cdot \nabla v\, dx + \int_{\partial\Omega} \mu(u_\mu - g)v\, ds = \int_\Omega fv dx \quad \forall v \in H^1(\Omega).$$

It is easy to see that this new boundary condition is approaching the original boundary condition as μ goes to infinity. Lions proved that the solution u_μ of the above problem converges to the solution u of the original problem as μ goes to infinity.

It can be proved that the convergence is achieved if μ is of the order of $h^{-1+\epsilon}$ for arbitrarily small $\epsilon > 0$. But rate of convergence of order in the energy norm is $h^{(2p+1)/3}$ when the penalty parameter μ is taken to be of the order of $h^{-(2p+1)/3}$. The lack of optimality in the order of convergence is a direct consequence of the lack of consistency of the weak formulation. Indeed, note that the exact solution u does not satisfy the above weak formulation. Instead, it satisfies

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds + \int_{\partial\Omega} \mu(u - g)v\, ds = \int_\Omega fv dx \quad \forall v \in H^1(\Omega), \tag{15.4}$$

or

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds + \int_{\partial\Omega} \mu uv\, ds = \int_\Omega fv dx + \int_{\partial\Omega} \mu g\, v\, ds \quad \forall v \in H^1(\Omega).$$

But the bilinear form on the right hand of the above equation is not symmetric. So naturally we add another term $-\int_{\partial\Omega}(u-g)\frac{\partial v}{\partial n}\, ds$, we get

$$\int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds - \int_{\partial\Omega} (u-g)\frac{\partial v}{\partial n}\, ds + \int_{\partial\Omega} \mu(u - g)v\, ds = \int_\Omega fv dx,$$

or

$$B(u, v) = \int_\Omega fv dx + \int_{\partial\Omega} \mu g\, v\, ds - \int_{\partial\Omega} g\frac{\partial v}{\partial n}\, ds,$$

where

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds - \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, ds + \int_{\partial\Omega} \mu uv\, ds.$$

for any weighting function μ. Note that the second term of the bilinear form B, which arises naturally from an integration by parts, ensures the consistency of the method. On the other hand, the third term renders the discrete problem symmetric and hence ensures the property of adjoint consistency. Finally, the last term penalizes the departure of the trace of the approximate solution from the Dirichlet data g and is necessary to guarantee stability. Nitsche proved that if μ is taken as η/h, where h is the element size and η is a sufficiently large constant, then the discrete solution converges to the exact solution with optimal order in H^1 and L^2.

Actually, we also can add term $\int_{\partial\Omega}(u-g)\frac{\partial v}{\partial n}\, ds$ into (15.4), which makes the symmetry of the bilinear form does not hold anymore, but the following new scheme is stable naturally.

$$B(u, v) = \int_\Omega fv dx + \int_{\partial\Omega} \mu g\, v\, ds + \int_{\partial\Omega} g\frac{\partial v}{\partial n}\, ds,$$

where

$$B(u, v) = \int_\Omega \nabla u \cdot \nabla v\, dx - \int_{\partial\Omega} \frac{\partial u}{\partial n} v\, ds + \int_{\partial\Omega} u\frac{\partial v}{\partial n}\, ds + \int_{\partial\Omega} \mu uv\, ds.$$

The IP methods arose from the observation that, just as Dirichlet boundary conditions could be imposed weakly instead of being built into the finite element space, so interelement continuity could be attained in a similar fashion.

On each element $K \in \mathcal{T}_h$, similarly, $u \in H^2(\Omega)$ satisfies the following identity

$$\int_K \nabla u \cdot \nabla v\, dx - \int_{\partial K} (\nabla u \cdot \mathbf{n})v\, ds + \int_{\partial K} \mu(u - u^{nb})v\, ds = \int_K f\, v\, dx$$

for all the $v \in H^2(\mathcal{T}_h)$. Here, u^{nb} is the value of the u on the edge of the neighbor element, and equal g on the boundary. Adding over all the elements $K \in \mathcal{T}_h$, we have

$$\int_\Omega \nabla u \cdot \nabla_h v \, dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nabla u \cdot \mathbf{n}) v \, ds + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu(u - u^{nb}) v \, ds = \int_\Omega f \, v \, dx \tag{15.5}$$

By the identity (23.62), we get

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nabla u \cdot \mathbf{n}) v \, ds = \sum_{e\in\mathcal{E}_h} \int_e \{\nabla u\} \cdot [v] \, ds + \sum_{e\in\mathcal{E}_h^i} \int_e [\nabla u]\{v\} \, ds = \int_{\mathcal{E}_h} \{\nabla u\} \cdot [v] \, ds.$$

The second term on the left vanishes due to the fact that $u \in H^2(\Omega)$, and so $[\nabla u] = 0$. Now, let us consider the term $\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu(u - u^{nb}) v \, ds$. It is easy to see that

$$\begin{aligned}
&\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu(u - u^{nb}) v \, ds \\
=& \sum_{e\in\mathcal{E}_h^i} \int_e \mu((u^+ - u^-)v^+ + (u^- - u^+)v^-) \, ds + \sum_{e\in\mathcal{E}_h^\partial} \int_e \mu(u - g) v \, ds \\
=& \sum_{e\in\mathcal{E}_h^i} \int_e \mu(u^+ - u^-)(v^+ - v^-)\mathbf{n} \cdot \mathbf{n} \, ds + \sum_{e\in\mathcal{E}_h^\partial} \int_e \mu(u - g) v \, ds \\
=& \sum_{e\in\mathcal{E}_h^i} \int_e \mu(u^+\mathbf{n}^+ + u^-\mathbf{n}^-) \cdot (v^+\mathbf{n}^+ + v^-\mathbf{n}^-) \, ds + \sum_{e\in\mathcal{E}_h^\partial} \int_e \mu(u - g) v \, ds \\
=& \int_{\mathcal{E}_h} \mu[u] \cdot [v] \, ds - \int_{\mathcal{E}_h^\partial} \mu g v \, ds
\end{aligned}$$

Taking the above two equalities into (23.63), we obtain

$$\int_\Omega \nabla u \cdot \nabla_h v \, dx - \int_{\mathcal{E}_h} \{\nabla u\} \cdot [v] \, ds + \int_{\mathcal{E}_h} \mu[u] \cdot [v] \, ds = \int_\Omega f \, v \, dx + \int_{\mathcal{E}_h^\partial} \mu g v \, ds$$

The bilinear form of left hand is not symmetric, so we add the term $-\int_{\mathcal{E}_h} [u] \cdot \{\nabla_h v\} \, ds$ on the left and $-\int_{\mathcal{E}_h^\partial} g(\nabla_h v \cdot \mathbf{n}) \, ds$ on the right, and finally get

$$B_h(u, v) = \int_\Omega f \, v \, dx + \int_{\mathcal{E}_h^\partial} \mu g v \, ds - \int_{\mathcal{E}_h^\partial} g(\nabla_h v \cdot \mathbf{n}) \, ds,$$

where the bilinear form is

$$B_h(u, v) := \int_\Omega \nabla_h u \cdot \nabla_h v \, dx - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [v] \, ds - \int_{\mathcal{E}_h} [u] \cdot \{\nabla_h v\} \, ds + \int_{\mathcal{E}_h} \frac{\eta}{h_e} [u] \cdot [v] \, ds$$

Here, we let the penalty constant $\mu = \eta h^{-1}$.

Similarly, we can change the sign of the third term of B_h to obtain Non-symmetric IP (NIPG) methods as follow.

$$B_h(u, v) = \int_\Omega f \, v \, dx + \int_{\mathcal{E}_h^\partial} \mu g v \, ds + \int_{\mathcal{E}_h^\partial} g(\nabla_h v \cdot \mathbf{n}) \, ds,$$

where the bilinear form is

$$B_h(u,v) := \int_\Omega \nabla_h u\cdot\nabla_h v\,dx - \int_{\mathcal{E}_h}\{\nabla_h u\}\cdot[v]\,ds + \int_{\mathcal{E}_h}[u]\cdot\{\nabla_h v\}\,ds + \int_{\mathcal{E}_h}\frac{\eta}{h_e}[u]\cdot[v]\,ds$$

This makes it possible to use spaces of discontinuous piecewise polynomials for solving second-order problems.

15.3 Numerical flux approach for DG method

In this section, we apply numerical flux approach to establish discontinuous Galerkin schemes, We now present some DG methods for the Poisson problem (15.1). To do it, we first rewrite the Poisson problem as the first order system

$$\mathbf{p} = \nabla u, \quad \text{in } \Omega \tag{15.6}$$

$$-\nabla\cdot\mathbf{p} = f \quad \text{in } \Omega, \tag{15.7}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{15.8}$$

We multiply the equations (15.99) and (15.100) by test functions $\mathbf{w}$ and v, respectively, and integrate on a subset $K\subset\Omega$. We get by integration by part,

$$\int_K \mathbf{p}\cdot\mathbf{w}\,dx = -\int_K u\nabla\cdot\mathbf{w}\,dx + \int_{\partial K} u\,\mathbf{n}_K\cdot\mathbf{w}\,ds \tag{15.9}$$

$$\int_K \mathbf{p}\cdot\nabla v\,dx = \int_K f\,v\,dx + \int_{\partial K}\mathbf{p}\cdot\, n_K\,v\,ds, \tag{15.10}$$

where n_K is the outward normal unit vector to ∂K. In above equations, we append subscript h on $\mathbf{p}$, u, ∇ and v, and use numerical traces $\widehat{u_h}$ and $\widehat{\mathbf{p}_h}$ to approximate u and $\mathbf{p}$ over element edges to get

$$\int_K \mathbf{p}_h\cdot\mathbf{w}_h\,dx = -\int_K u_h\,\nabla_h\cdot\mathbf{w}_h\,dx + \int_{\partial K}\widehat{u_h}\,n_K\cdot\mathbf{w}_h\,ds$$

$$\int_K \mathbf{p}_h\cdot\nabla_h\mathbf{v}_h\,dx = \int_K f\,v_h\,dx + \int_{\partial K}\widehat{\mathbf{p}_h}\cdot\, n_K\,v_h\,ds.$$

Then we add over all the elements to obtain

$$\int_\Omega \mathbf{p}_h\cdot\mathbf{w}_h\,dx = -\int_\Omega u_h\,\nabla_h\cdot\mathbf{w}_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{u_h}\,n_K\cdot\mathbf{w}_h\,ds \qquad \forall\,\mathbf{w}_h\in W_h, \tag{15.11}$$

$$\int_\Omega \mathbf{p}_h\cdot\nabla_h v_h\,dx = \int_\Omega f\,v_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{\mathbf{p}_h}\cdot\, n_K\,v_h\,ds \qquad \forall\,v_h\in V_h, \tag{15.12}$$

for all $(\mathbf{w}_h, v_h)\in W_h\times V_h$ and all $K\in\mathcal{T}_h$. The numerical traces $\widehat{\mathbf{p}_h}$ and $\widehat{u_h}$ will be selected to guarantee consistency and stability of the above scheme. Different choices of the numerical fluxes leads to different DG methods.

To derive a new formulation which does not rely on $\mathbf{p}_h$ explicitly, using (15.2) and (23.62), we have from (15.106) and (15.107) that

$$\int_\Omega \mathbf{p}_h\cdot\mathbf{w}_h\,dx = \int_\Omega \nabla_h u_h\cdot\mathbf{w}_h\,dx + \sum_{e\in\mathcal{E}_h}\int_e[\widehat{u_h}-u_h]\cdot\{\mathbf{w}_h\}\,ds + \sum_{e\in\mathcal{E}_h^i}\int_e\{\widehat{u_h}-u_h\}[\mathbf{w}_h]\,ds, \tag{15.13}$$

$$\int_\Omega \mathbf{p}_h\cdot\nabla_h v_h\,dx = \int_\Omega f\,v_h\,dx + \sum_{e\in\mathcal{E}_h}\int_e[v_h]\cdot\{\widehat{\mathbf{p}_h}\}\,ds + \sum_{e\in\mathcal{E}_h^i}\int_e\{v_h\}[\widehat{\mathbf{p}_h}]\,ds. \tag{15.14}$$

Choosing $\mathbf{w}_h = \nabla_h v_h$ in (15.108), we have

$$\int_\Omega \mathbf{p}_h \cdot \nabla_h v_h \, dx = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx + \sum_{e \in \mathcal{E}_h} \int_e [\widehat{u_h} - u_h] \cdot \{\nabla_h v_h\} \, ds + \sum_{e \in \mathcal{E}_h^i} \int_e \{\widehat{u_h} - u_h\}[\nabla_h v_h] \, ds.$$

The combination of the last equation and (15.109) yields

$$B_h(u_h, v_h) = \int_\Omega f \, v_h \, dx. \tag{15.15}$$

where

$$\begin{aligned} B_h(u_h, v_h) = & \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx + \sum_{e \in \mathcal{E}_h} \int_e [\widehat{u_h} - u_h] \cdot \{\nabla_h v_h\} \, ds + \sum_{e \in \mathcal{E}_h^i} \int_e \{\widehat{u_h} - u_h\}[\nabla_h v_h] \, ds \\ & - \sum_{e \in \mathcal{E}_h} \int_e [v_h] \cdot \{\widehat{\mathbf{p}_h}\} \, ds - \sum_{e \in \mathcal{E}_h^i} \int_e \{v_h\}[\widehat{\mathbf{p}_h}] \, ds. \end{aligned} \tag{15.16}$$

We can get DG methods from (15.114) by proper choices of numerical traces $\widehat{\mathbf{p}_h}$ and $\widehat{u_h}$. There are three principles for choosing appropriate numerical traces. Conservation requires the numerical traces to be single-valued over all edges; consistency of the numerical traces requires $\widehat{u_h}(u) = u|_{\mathcal{E}_h}$ and $\widehat{\mathbf{p}_h}(u, \mathbf{p}) = \mathbf{p}|_{\mathcal{E}_h}$; stability is not easily ensured and it is usual to add a suitable penalty term (stability term) to guarantee it. We will introduce five consistent and stable DG methods.

The first example is IP method. Taking

$$\begin{cases} \widehat{u_h} = \{u_h\} \quad \text{on } \mathcal{E}_h^i, \qquad \widehat{u_h} = 0 \quad \text{on } \partial\Omega, \\ \widehat{\mathbf{p}_h} = \{\nabla_h u_h\} - \dfrac{\eta}{h_e}[u_h] \quad \text{on } \mathcal{E}_h, \end{cases}$$

where the function η equals a constant η_e on each $e \in \mathcal{E}_h$, with $\{\eta_e\}_{e \in \mathcal{E}_h}$ having a uniform positive bound from above and below. It is easy to see that the above numerical fluxes are consistent and single valued, which means they are conservative.

We obtain from (15.114) and (15.116) that

$$B_{1,h}^{(1)}(u_h, v_h) = \int_\Omega f \, v_h \, dx \tag{15.17}$$

where

$$B_{1,h}^{(1)}(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] \, ds \tag{15.18}$$

$$- \int_{\mathcal{E}_h} [u_h] \cdot \{\nabla_h v_h\} \, ds + \int_{\mathcal{E}_h} \frac{\eta}{h_e} [u_h] \cdot [v_h] \, ds. \tag{15.19}$$

The term $\int_{\mathcal{E}_h} \eta h_e^{-1} [u_h] \cdot [v_h] \, ds$ is the penalty term. This is how to derive IP method through numerical flux approach, and we get the same scheme with the one by adding penalty on the element edges in the previous section.

For the analysis convenience, we will introduce the following lifting operator, $r : [L^2(\mathcal{E}_h)]^2 \rightarrow W_h$, , $r_e : [L^2(e)]^2 \rightarrow W_h$ and $l : L^2(\mathcal{E}_h^i) \rightarrow W_h$ are lifting operators defined by

$$\int_\Omega r(\mathbf{q})\cdot\mathbf{w}_h\,dx = -\int_{\mathcal{E}_h}\mathbf{q}\cdot\{\mathbf{w}_h\}\,ds, \quad \forall\,\mathbf{w}_h\in W_h, \tag{15.20}$$

$$\int_\Omega r_e(\mathbf{q})\cdot\mathbf{w}_h\,dx = -\int_e\mathbf{q}\cdot\{\mathbf{w}_h\}\,ds, \quad \forall\,\mathbf{w}_h\in W_h, \tag{15.21}$$

$$\int_\Omega l(v)\cdot\mathbf{w}_h\,dx = -\int_{\mathcal{E}_h^i} v[\mathbf{w}_h]\,ds \quad \forall\,\mathbf{w}_h\in W_h. \tag{15.22}$$

With the lift operator r, we can rewrite $B^{(1)}_{1,h}$ as

$$B^{(1)}_{2,h}(u_h,v_h) := \int_\Omega\nabla_h u_h\cdot\nabla_h v_h\,dx + \int_\Omega\nabla_h u_h\cdot r([v_h])\,dx \tag{15.23}$$

$$+\int_\Omega r([u_h])\cdot\nabla_h v_h\,dx + \int_{\mathcal{E}_h}\frac{\eta}{h_e}[u_h]\cdot[v_h]\,ds. \tag{15.24}$$

Note that (15.19) and (15.24) are equivalent on V_h, implying that either one can be used to define the numerical solution u_h. In this paper, we give a priori error estimate for the first formula (15.19). Because (15.19) and (15.24) are equivalent on V_h, we will prove the stability for the second formula $B^{(1)}_{2,h}$ on V_h, which guarantees the stability for the first formulation $B^{(1)}_{1,h}$ on V_h. This comment is valid for the other DG methods to be introduced later.

By changing the sign of the second term in the bilinear form $B^{(1)}_{1,h}$, we can give a non-symmetric interior penalty (NIPG) formulation,

$$\begin{aligned} B^{(2)}_{1,h}(u_h,v_h) := {} & \int_\Omega\nabla_h u_h\cdot\nabla_h v_h\,dx - \int_{\mathcal{E}_h}\{\nabla_h u_h\}\cdot[v_h]\,ds \\ & + \int_{\mathcal{E}_h}[u_h]\cdot\{\nabla_h v_h\}\,ds + \int_{\mathcal{E}_h}\frac{\eta}{h_e}[u_h]\cdot[v_h]\,ds, \end{aligned}$$

or equivalently,

$$\begin{aligned} B^{(2)}_{2,h}(u_h,v_h) := {} & \int_\Omega\nabla_h u_h\cdot\nabla_h v_h\,dx + \int_\Omega\nabla_h u_h\cdot r([v_h])\,dx \\ & - \int_\Omega r([u_h])\cdot\nabla_h v_h\,dx + \int_{\mathcal{E}_h}\frac{\eta}{h_e}[u_h]\cdot[v_h]\,ds. \end{aligned}$$

It is not hard to choose proper numerical fluxes to satisfy consistency and conservation, but the stability of the DG scheme is not easy to guarantee. Let us consider to construct a DG method by satisfying scheme stability. To do this, we multiply (15.99) and (15.100) by p and u respectively, then integrate over Ω to get

$$\begin{aligned} \int_\Omega|\mathbf{p}|^2\,dx &= \int_\Omega\mathbf{p}\cdot\nabla u\,dx, \\ -\int_\Omega\nabla\cdot\mathbf{p}\,u\,dx &= \int_\Omega f\,u\,dx. \end{aligned}$$

Using the boundary condition that $u = 0$ on $\partial\Omega$ and integration by part, we add the above two equations to obtain

$$\int_\Omega|\mathbf{p}|^2\,dx = \int_\Omega f\,u\,dx.$$

This equation can be understood as energy conservation. To find a stable scheme for DG method, we mimic this procedure as follow. By taking $w_h = \mathbf{p}_h$ in (15.106) and $v_h = u_h$ in (15.107), we obtain

$$\int_\Omega |\mathbf{p}_h|^2\,dx = -\int_\Omega u_h\,\nabla_h\cdot\mathbf{p}_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{u_h}\,n_K\cdot\mathbf{p}_h\,ds,$$

$$\int_\Omega \mathbf{p}_h\cdot\nabla_h u_h\,dx = \int_\Omega f\,u_h\,dx + \sum_{K\in\mathcal{T}_h}\int_{\partial K}\widehat{\mathbf{p}_h}\cdot\,n_K\,u_h\,ds.$$

Doing integration by part to the left term of the second equation, then combine the above two equations to get

$$\int_\Omega |\mathbf{p}_h|^2\,dx + \Lambda_h = \int_\Omega f\,u_h\,dx.$$

where

$$\Lambda_h = \sum_{K\in\mathcal{T}_h}\int_{\partial K}(u_h\,n_K\cdot\mathbf{p}_h - \widehat{u_h}\,n_K\cdot\mathbf{p}_h - \widehat{\mathbf{p}_h}\cdot\,n_K\,u_h)\,ds$$

Let us find consistent numerical fluxes $\widehat{u_h}$ and $\widehat{\mathbf{p}_h}$ to make the Λ_h nonnegative. Note that conservation requires the numerical traces to be single-valued over all edges, so $[\widehat{u_h}] = \widehat{u_h}$ and $[\widehat{\mathbf{p}_h}] = \widehat{\mathbf{p}_h}$. By some algebra manipulations, we have

$$\begin{aligned}\Lambda_h &= \sum_{e\in\mathcal{E}_h}\int_e[u_h\mathbf{p}_h - \widehat{u_h}\mathbf{p}_h - u_h\widehat{\mathbf{p}_h}]\,ds\\ &= \sum_{e\in\mathcal{E}_h}\int_e[u_h\mathbf{p}_h] - \widehat{u_h}[\mathbf{p}_h] - [u_h]\cdot\widehat{\mathbf{p}_h}\,ds\\ &= \sum_{e\in\mathcal{E}_h}\int_e[u_h\mathbf{p}_h] - \widehat{u_h}[\mathbf{p}_h] - [u_h]\cdot\widehat{\mathbf{p}_h}\,ds\\ &= \sum_{e\in\mathcal{E}_h}\int_e(\{u_h\} - \widehat{u_h})[\mathbf{p}_h] + [u_h]\cdot(\{\mathbf{p}_h\} - \widehat{\mathbf{p}_h})\,ds + \sum_{e\in\mathcal{E}_h^\partial}\int_e(u_h(\mathbf{p}_h - \widehat{\mathbf{p}_h})\cdot n - \widehat{u_h}\mathbf{p}_h\cdot n)\,ds\end{aligned}$$

In the last equality, we use the identity

$$[u_h\mathbf{p}_h] = \{u_h\}[\mathbf{p}_h] + [u_h]\cdot\{\mathbf{p}_h\} \text{ on } e\in\mathcal{E}_h^i, \quad \text{and} \quad [u_h\mathbf{p}_h] = \{u_h\}[\mathbf{p}_h] = u_h\mathbf{p}_h\mathbf{n} \text{ on } e\in\mathcal{E}_h^\partial.$$

If we take on the interior edges $e\in\mathcal{E}_h^i$,

$$\widehat{\mathbf{p}_h} = \{\mathbf{p}_h\} - C_{11}[u_h] + C_{12}[\mathbf{p}_h], \quad \widehat{u_h} = \{u_h\} - C_{12}[u_h] - C_{22}[\mathbf{p}_h],$$

and on the boundary edge

$$\widehat{\mathbf{p}_h} = \mathbf{p}_h - C_{11}u_h\mathbf{n}, \quad \widehat{u_h} = 0,$$

we have

$$\Lambda_h = \sum_{e\in\mathcal{E}_h^i}\int_e(C_{22}[p_h]^2 + C_{11}[u_h]^2)\,ds + \sum_{e\in\mathcal{E}_h^\partial}\int_e C_{11}u_h^2\,ds \geq 0,$$

provided $C_{11}\geq 0$ and $C_{22}\geq 0$.

To avoid the present of $\mathbf{p}_h$ in the DG formulation, we need the following identity.

$$\mathbf{p}_h = \nabla_h u_h - r([\widehat{u_h} - u_h]) - l(\{\widehat{u_h} - u_h\}). \tag{15.25}$$

(The proof of this identity is left as homework.)

If we choose $C_{11} = \boldsymbol{\eta} h^{-1}$, $C_{12} = \boldsymbol{\beta}$, and $C_{22} = 0$, i.e.,

$$\begin{cases} \widehat{u_h} = \{u_h\} - \boldsymbol{\beta} \cdot [u_h] \quad \text{on } \mathcal{E}_h^i, \qquad \widehat{u_h} = 0 \quad \text{on } \partial\Omega, \\ \widehat{\mathbf{p}_h} = \{\mathbf{p}_h\} + \beta[\mathbf{p}_h] - \dfrac{\eta}{h_e}[u_h] \quad \text{on } \mathcal{E}_h, \end{cases}$$

and apply the relation (15.25), then we obtain the following local DG (LDG) method.

$$\begin{aligned} B_{1,h}^{(3)}(u_h, v_h) :=& \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] \, ds - \int_{\mathcal{E}_h} [u_h] \cdot \{\nabla_h v_h\} \, ds \\ &- \int_{\mathcal{E}_h^i} (\boldsymbol{\beta} \cdot [u_h][\nabla_h v_h] + [\nabla_h u_h] \boldsymbol{\beta} \cdot [v_h]) \, ds \\ &+ \int_\Omega (r([u_h]) + l(\boldsymbol{\beta} \cdot [u_h]))\,(r([v_h]) + l(\boldsymbol{\beta} \cdot [v_h])) \, dx + \int_{\mathcal{E}_h} \frac{\eta}{h_e} [u_h] \cdot [v_h] \, ds, \end{aligned}$$

Here, we have $\mathbf{p}_h = \nabla_h u_h + r([u_h]) + l([u_h])$ due to $\widehat{u_h} = \{u_h\} - \boldsymbol{\beta} \cdot [u_h]$ and $[\{u_h\}] = 0$, $[[u_h]] = 0$, $\{[u_h]\} = [u_h]$, $\{\{u_h\}\} = \{u_h\}$.

Using the local lifting operator r_e, we can give the fourth example. Taking

$$\begin{cases} \widehat{u_h} = \{u_h\} \quad \text{on } \mathcal{E}_h^i, \qquad \widehat{u_h} = 0 \quad \text{on } \partial\Omega, \\ \widehat{p_h} = \{p_h\} - \eta_e r_e([u_h]) \quad \text{on } \mathcal{E}_h, \end{cases}$$

from (15.114), we get

$$\begin{aligned} B_{1,h}^{(4)}(u_h, v_h) :=& \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] \, ds - \int_{\mathcal{E}_h} [u_h] \cdot \{\nabla_h v_h\} \, ds \\ &+ \int_\Omega r([u_h]) \cdot r([v_h]) \, dx + \sum_{e \in \mathcal{E}_h} \int_\Omega \eta_e r_e([u_h]) \cdot r_e([v_h]) \, dx, \end{aligned}$$

or equivalently,

$$B_{2,h}^{(4)}(u_h, v_h) := \int_\Omega (\nabla_h u_h + r([u_h])) \cdot (\nabla_h v_h + r([v_h]) \, dx + \sum_{e \in \mathcal{E}_h} \int_\Omega \eta_e r_e([u_h]) \cdot r_e([v_h]) \, dx.$$

which is the method of Brezzi et al.

With the choice

$$\begin{cases} \widehat{u_h} = \{u_h\} \quad \text{on } \mathcal{E}_h^i, \qquad \widehat{u_h} = 0 \quad \text{on } \partial\Omega, \\ \widehat{p_h} = \{\nabla_h u_h\} - \eta_e r_e([u_h]) \quad \text{on } \mathcal{E}_h, \end{cases}$$

we obtain a DG formulation of Bassi et al.,

$$\begin{aligned} B_{1,h}^{(5)}(u_h, v_h) :=& \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] \, ds - \int_{\mathcal{E}_h} [u_h] \cdot \{\nabla_h v_h\} \, ds \\ &+ \sum_{e \in \mathcal{E}_h} \int_\Omega \eta_e r_e([u_h]) \cdot r_e([v_h]) \, dx, \end{aligned}$$

or equivalently,

$$B_{2,h}^{(5)}(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx + \int_\Omega \nabla_h u_h \cdot r([v_h]) \, dx + \int_\Omega r([u_h]) \cdot \nabla_h v_h \, dx$$
$$+ \sum_{e \in \mathcal{E}_h} \int_\Omega \eta_e r_e([u_h]) \cdot r_e([v_h]) \, dx.$$

Let us make a summary and also introduce the numerical fluxes of other four DG methods in the following table.

Table Some DG schemes and their numerical fluxes.

Schemes	$\widehat{u_h}$	$\widehat{\mathbf{p}_h}$
1. IP	$\{u_h\}$	$\{\nabla_h u_h\} - \eta h^{-1}[u_h]$
2. NIPG	$\{u_h\} + \mathbf{n}_K \cdot [u_h]$	$\{\nabla_h u_h\} - \eta h^{-1}[u_h]$
3. LDG	$\{u_h\} - \boldsymbol{\beta} \cdot [u_h]$	$\{\mathbf{p}_h\} + \beta[\mathbf{p}_h] - \eta h^{-1}[u_h]$
4. Brezzi et al.	$\{u_h\}$	$\{\mathbf{p}_h\} - \eta_e r_e([u_h])$
5. Bassi et al.	$\{u_h\}$	$\{\nabla_h u_h\} - \eta_e r_e([u_h])$
6. BabuVska-Zlámal	$(u_h\|_K)\|_{\partial K}$	$-\eta h^{-1}[u_h]$
7. Brezzi et al.(Inconsistent)	$(u_h\|_K)\|_{\partial K}$	$-\eta_e r_e([u_h])$
8. Baumann-Oden	$\{u_h\} + \mathbf{n}_K \cdot [u_h]$	$\{\nabla_h u_h\}$
9. Bassi-Rebay	$\{u_h\}$	$\{\mathbf{p}_h\}$

Note that in the above table, the first five DG schemes are consistent and stable; The next two DG methods are inconsistent, and the last two DG scheme are unstable. Hence, the first five DG methods are preferable. In the following table, we give the bilinear form of the DG schemes. We use the shorter notation $(w, v)_\Omega$, $\langle w, v\rangle_{\mathcal{E}_h}$, and $\langle w, v\rangle_{\mathcal{E}_h^i}$ instead of $\int_\Omega wv \, dx$, $\int_{\mathcal{E}_h} wv \, ds$, and $\int_{\mathcal{E}_h^i} wv \, ds$, and $\alpha^j = \int_{\mathcal{E}_h} \frac{\eta}{h_e}[u_h] \cdot [v_h] \, ds$, $\alpha^r = \sum_{e \in \mathcal{E}_h} \int_\Omega \eta_e r_e([u_h]) \cdot r_e([v_h]) \, dx$.

Table Bilinear forms of the DG methods

Methods	Bilinear forms $B_h^{(j)} - \int_\Omega \nabla_h w \cdot \nabla_h v \, dx$
1. IP	$-\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} + \alpha^j$
2. NIPG	$\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} + \alpha^j$
3. LDG	$-\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} - \langle \beta \cdot [w], [\nabla_h v]\rangle_{\mathcal{E}_h^i} + \alpha^j$ $-\langle [\nabla_h w], \beta \cdot [v]\rangle_{\mathcal{E}_h^i} + (r([w]) + l(\beta \cdot [w]), r([v]) + l(\beta \cdot [v]))_\Omega$
4. Brezzi et al.	$-\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} + (r([w]), r([v]))_\Omega + \alpha^r$
5. Bassi et al.	$-\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} + \alpha^r$
6. BabuVska-Zlámal	α^j
7. Brezzi et al. (Incon)	α^r
8. Baumann-Oden	$\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h}$
9. Bassi-Rebay	$-\langle [w], \{\nabla_h v\}\rangle_{\mathcal{E}_h} - \langle \{\nabla_h w\}, [v]\rangle_{\mathcal{E}_h} + (r([w]), r([v]))_\Omega$

15.4 Consistency, boundedness and stability

Lemma 109 (Consistency). *Assume the numerical traces are consistent, i.e.,* $\widehat{u_h}(u) = u|_{\mathcal{E}_h}$ *and* $\widehat{p_h}(u, p) = p|_{\mathcal{E}_h}$. *Then the DG schemes with* $B_h(w, v) = B_h^{(j)}(w, v)$, $i = 1 \cdots 5, 8, 9$ *are consistent , that is*

$$B_h(u, v_h) = \int_\Omega f \, v_h \, dx \quad \forall \, v_h \in H^2(\mathcal{T}_h), \tag{15.26}$$

Proof. Let take IP method as an example to show the proof, and the similar argument can be applied to the rest DG schemes. (The proof for result schemes are exercises or homework.)

Note that $[u] = 0$ on $\mathcal{E}_h$, by (15.19) and (23.62), we have

$$B^{(1)}_{1,h}(u, v_h) := \int_\Omega \nabla u \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [v_h] \, ds \tag{15.27}$$

$$= \int_\Omega -\Delta u \, v_h \, dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nabla u \cdot \mathbf{n}_K v_h \, ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [v_h] \, ds$$

$$= \int_\Omega f \, v_h \, dx,$$

which completes the proof. □

With the consistency (16.4), and the DG scheme (15.114), it is easy to see that the Galerkin orthogonality holds

$$B_h(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h, \tag{15.28}$$

To consider the boundedness and stability of the bilinear form B_h, let $V(h) = V_h + H^2(\Omega) \cap H^1_0(\Omega) \subset H^2(\mathcal{T}_h)$, and define seminorms and norm for $v \in V(h)$ by the following relations:

$$|v|^2_{1,h} = \sum_{K\in\mathcal{T}_h} |v|^2_{1,K}, \quad |v|^2_{1,*} = \sum_{e\in\mathcal{E}_h} h_e^{-1} \|[v]\|^2_{0,e}$$

$$\|v\|^2_* = |v|^2_{1,h} + \sum_{K\in\mathcal{T}_h} h_K^2 |v|^2_{2,K} + |v|^2_{1,*}. \tag{15.29}$$

That (16.6) defines a norm can be seen from the next inequality (Poincare type inequality):

$$\|v\|_0 \le C(|v|^2_{1,h} + |v|^2_{1,*})^{1/2} \le C\,\|v\|_* \quad \forall\, v \in V(h). \tag{15.30}$$

Notice that the norm $\|v\|_*$ is the good to obtain boundedness of the bilinear form B_h, and the weaker norm $\|v\|_w = (|v|^2_{1,h} + |v|^2_{1,*})^{1/2}$ is the natural choice for analyzing the stability of DG methods.

Lemma 110. *Let $e \in \partial K$, then for any $v \in H^1(K)$, we have*

$$\|v\|^2_{0,e} \le C(h_K^{-1}\|v\|^2_{0,K} + h_K |v|^2_{1,K}). \tag{15.31}$$

Proof. Given K, let $\mathbf{n}$ be the unit outward normal vector on ∂K. Let $\boldsymbol{\rho}$ be vector field on $K \cup \partial K$ such that $\boldsymbol{\rho}\cdot\mathbf{n} \ge c_0$, where $c_0 > 0$. Then,

$$c_0 \int_{\partial K} v^2 \, ds \le \int_{\partial K} v^2 \boldsymbol{\rho}\cdot\mathbf{n} \, ds = -\int_K \nabla\cdot(v^2\boldsymbol{\rho}) \, dx$$

$$= -\int_K \nabla\cdot(v\nabla v)\cdot\boldsymbol{\rho} \, dx - \int_K v^2 \nabla\cdot\boldsymbol{\rho} \, dx$$

$$\le \|\boldsymbol{\rho}\|_{\infty,K} \int_K |v||\nabla v| \, dx + \|\nabla\cdot\boldsymbol{\rho}\|_{\infty,K} \int_K v^2 \, dx$$

Let (x_0, y_0) be the center of the inscribed circle of element K, then choose $\boldsymbol{\rho} = (x - x_0, y - y_0)$. It is easy to check that $\|\boldsymbol{\rho}\|_{\infty,D} \le h_K$, $\|\nabla\cdot\boldsymbol{\rho}\|_{\infty,D} \le 2$, and $c_0 \ge r$, where r is the radius of the inscribed circle of element K. Because the element K is shape regular, i.e., h_K can be bounded by r, then by Schwarz inequality, we finish the proof. □

With this trace inequality, we have the following results.

Lemma 111 (Boundedness). *For* $1 \le j \le 9$, $B_h = B_h^{(j)}$ *satisfies*

$$B_h(u,v) \le C_b \|u\|_* \, \|v\|_* \quad \forall\, u,v \in V(h), \tag{15.32}$$

where C_b *is a positive constant depending on the angle condition, the polynomial degree, an upper bound on the edge-dependent penalty parameter* η *for the methods that contain the penalty term* α^j *or* α^r *and, in the case of the LDG method, an upper bound for the coefficient* β.

Proof. Let us take IP method as an example to show the proof.
It is ease to see that

$$\int_\Omega \nabla_h w \cdot \nabla_h v \, dx \le |w|_{1,h} |v|_{1,h}$$

and

$$\int_{\mathcal{E}_h} \frac{\eta}{h_e} [w] \cdot [v_h] \, ds \le C |w|_{1,*} |v|_{1,*}$$

Let us consider the term $\int_{\mathcal{E}_h} \{\nabla_h w\} \cdot [v] \, ds$ and another term $\int_{\mathcal{E}_h} [w] \cdot \{\nabla_h v_h\} \, ds$ can be analyzed in the same way.

$$\begin{aligned} \int_{\mathcal{E}_h} \{\nabla_h w\} \cdot [v] \, ds &= \sum_{\mathcal{E}_h} \int_e (h_e^{1/2} \{\nabla_h w\}) \cdot (h_e^{-1/2} [v]) \, ds \\ &\le C \left[\sum_{K \in \mathcal{T}_h} (|w|_{1,k}^2 + h_K^2 |u|_{2,k}^2) \right]^{1/2} \left[\sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e |[v]|^2 ds \right]^{1/2} \\ &\le C \|w\|_* |v|_* \end{aligned}$$

In the above inequalities, we apply the trace inequality (15.134). So we prove

$$B_{1,h}^{(1)}(w,v) \le C_b \|w\|_* \|v\|_*$$

□

For the stability, we have the following results.

Lemma 112 (Stability). *For* $1 \le j \le 7$, $B_h = B_h^{(j)}$ *satisfies*

$$B_h(v,v) \ge C_s \|v\|_*^2 \quad \forall\, v \in V_h, \tag{15.33}$$

if $\eta_0 = \inf_e \eta_e > 0$ *for the methods with* $j = 2,3,4,6,7$, $\eta_0 > 3$ *for the method with* $j = 5$, η_0 *is large enough for IP method* ($j = 1$), *where* C_s *is a positive constant depending on the angle condition, the polynomial degree, a bound on the edge-dependent penalty parameter* η *and, in the case of the LDG method, a bound for the coefficient* β.

Proof. Again, let us consider IP method as an example to prove this lemma.
From the proof of the boundedness, we know that

$$\int_{\mathcal{E}_h} \{\nabla_h v\} \cdot [v] \, ds \le C \|v\|_* |v|_{1,*}$$

Then

$$\begin{aligned} B_{1,h}^{(1)}(v,v) &\ge |v|_{1,h}^2 + \eta_0 |v|_{1,*}^2 - C\|v\|_* |v|_{1,*} \ge |v|_{1,h}^2 + \eta_0 |v|_{1,*}^2 - C(\epsilon \|v\|_*^2 + \frac{|v|_{1,*}^2}{4\epsilon}) \\ &= |v|_{1,h}^2 + (\eta_0 - \frac{C}{4\epsilon}) |v|_{1,*}^2 - C\epsilon \|v\|_*^2, \end{aligned}$$

where the $0 < \epsilon < 1$ is an arbitrary constant number. If we let ϵ is very very small, then the last term can be ignored. Hence,

$$B^{(1)}_{1,h}(v,v) \geq C_s(|v|^2_{1,h} + |v|^2_{1,*}) = C_s\|v\|^2_w, \tag{15.34}$$

If we choose η_0 be large enough. □

Notice that (16.11) only claims the coercivity of the bilinear form B_h on V_h. Lack of coercivity of B_h on V is one source of difficulty in studying the DG methods.

15.5 Approximation and error estimates

We now turn to error estimations for the DG methods. Write the error as

$$e = u - u_h = (u - u_I) + (u_I - u_h),$$

where $u_I \in V_h$ is a suitable interpolant of the exact solution. If u_I is chosen to be the usual continuous piecewise polynomial interpolant, then the jumps of $(u - u_I)$ will be zero at the interelement boundaries. Then

$$\|u - u_I\|^2_* = |u - u_I|^2_{1,h} + \sum_{K\in\mathcal{T}_h} h^2_K|u - u_I|^2_{2,K} + \sum_{e\in\mathcal{E}^\partial_h} h^{-1}_e\|[u - u_I]\|^2_{0,e} \leq C^2_a h^{2p}|u|^2_{p+1,\Omega}. \tag{15.35}$$

To analyze the method of Baumann-Oden ($j = 8$) and extend the analysis to nonconforming meshes, it is convenient to take an interpolant u_I which is discontinuous across the interelement boundaries. We just require the local approximation property

$$|u - u_I|_{s,K} \leq Ch^{p+1-s}_K|u|_{p+1,K};$$

then for the global approximation error, we have

$$\|u - u_I\|_* \leq C_a h^p|u|_{p+1,\Omega}. \tag{15.36}$$

15.5.1 Error Analysis for stable and consistent DG methods

Theorem 109 (Strang's Lemma). *Let u and u_h be the solutions of* (15.1) *and* (15.114) *respectively. Assume $u \in H^{p+1}(\Omega)$. Then for all the DG methods, we have*

$$\|u - u_h\|_* \leq (1 + \frac{C_b}{C_s}) \inf_{v_h\in V_h} \|u - v_h\|_* + \frac{1}{C_s} \sup_{w_h\in V_h} \frac{|B_h(u, w_h) - (f, w_h)|}{\|w_h\|_*}. \tag{15.37}$$

Proof. Let v_h be the usual continuous piecewise linear interpolant of u. Recall the boundedness and stability of the bilinear form B_h. We have

$$\begin{aligned} C_s\|v_h - u_h\|^2_* &\leq B_h(v_h - u_h, v_h - u_h) \\ &= B_h(v_h - u, v_h - u_h) + B_h(u - u_h, v_h - u_h), \\ &\leq C_b\|v_h - u\|_*\,\|v_h - u_h\|_* + B_h(u - u_h, v_h - u_h) \end{aligned}$$

So

$$\|v_h - u_h\|_* \leq \frac{C_b}{C_s}\|v_h - u\|_* + \frac{1}{C_s}\frac{B_h(u - u_h, v_h - u_h)}{\|v_h - u_h\|_*} \tag{15.38}$$

Furthermore,

$$\begin{aligned}\|u-u_h\|_* &\le \|u-v_h\|_* + \|v_h-u_h\|_*\\ &\le (1+\frac{C_b}{C_s})\|u-v_h\|_* + \frac{1}{C_s}\frac{B_h(u-u_h, v_h-u_h)}{\|v_h-u_h\|_*}\\ &\le (1+\frac{C_b}{C_s})\|u-v_h\|_* + \frac{1}{C_s}\sup_{w_h\in V_h}\frac{|B_h(u-u_h,w_h)|}{\|w_h\|_*}\end{aligned}$$

which finishes the proof. □

Note that if the consistent error

$$\inf_{v_h\in V_h}\|u-v_h\|_* + \sup_{w_h\in V_h}\frac{|B_h(u,w_h)-(f,w_h)|}{\|w_h\|_*} \lesssim h^p,$$

then we get

$$\|u-u_h\|_* \lesssim h^p. \tag{15.39}$$

Theorem 110. *Let u and u_h be the solutions of* (15.1) *and* (15.114) *respectively. Assume $u\in H^{p+1}(\Omega)$. Then for the DG methods with $j=1,\cdots,5$, we have*

$$\|u-u_h\| \le Ch^p, \tag{15.40}$$

where C is a positive constant that depends on $|u|_{p+1}$, the angle condition, a bound on the edge-dependent penalty parameter η and, in the case of the LDG method, a bound for the coefficient β.

Proof. Let u_I be the usual continuous piecewise linear interpolant of u. Recall the boundedness and stability of the bilinear form B_h. We have

$$C_s\|u_I-u_h\|_*^2 \le B_h(u_I-u_h, u_I-u_h) \equiv T_1+T_2, \tag{15.41}$$

where

$$\begin{aligned}T_1 &= B_h(u_I-u, u_I-u_h),\\ T_2 &= B_h(u-u_h, u_I-u_h) = 0.\end{aligned}$$

We bound T_1 as follows:

$$T_1 \le C_b\|u_I-u\|_*\,\|u_I-u_h\|_* \tag{15.42}$$

Then we get

$$\|u_I-u_h\|_* \le C_b/C_s\|u_I-u\|_* \le Ch^p. \tag{15.43}$$

Then we prove the (16.13). □

15.5.2 Inconsistent DG methods

For the DG methods of BabuVska–Zlámal ($j=6$) and Brezzi et al. ($j=7$), we can not get (15.130) as in the proof of Lemma 117, because for these two methods, the bilinear forms do not contain the term $\int_{\mathcal{E}_h}\{\nabla_h w\}\cdot[v]\,ds$. Instead of (15.130), we have

$$B_h(u, u_I-u_h) = -\int_\Omega \Delta u(u_I-u_h)\,dx + \int_{\mathcal{E}_h}\{\nabla_h u\}\cdot[u_I-u_h]\,ds, \tag{15.44}$$

implying that we have to bound the terms $\int_{\mathcal{E}_h}\{\nabla_h u\}\cdot[u_I - u_h]\,ds$. Even though B_h of the two pure penalty methods are stable and bounded for the norm $\|\cdot\|_*$ defined in (16.6), the difficulty is in giving good estimates of $\int_{\mathcal{E}_h}\{\nabla_h u\}\cdot[u_I - u]\,ds$ dependent on the norm $\|\cdot\|_*$. The superpenalties can be used to reduce the influence that (15.130) does not hold true for these two methods. For the method of BabuVska-Zlámal ($j = 6$), take the penalty term as

$$\alpha^j(u,v) = \sum_{e\in\mathcal{E}_h}\int_e \eta_e h_e^{-2p-1}[u]\cdot[v]\,ds.$$

The corresponding bilinear form is bounded with respect to the norm $\|\cdot\|_s$ defined by

$$\|v\|_s^2 = |v|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2|v|_{2,K}^2 + \alpha^j(v,v). \tag{15.45}$$

Then we have, for all $u, v\in V(h)$,

$$\begin{aligned}\sum_{e\in\mathcal{E}_h}\int_e\{\nabla_h u\}\cdot[v]\,ds &= \sum_{e\in\mathcal{E}_h}\int_e (h_e^{2p+1})^{1/2}\{\nabla_h u\}\cdot[v](h_e^{-2p-1})^{1/2}\,ds\\ &\le C\|v\|_s\left(\sum_{e\in\mathcal{E}_h} h_e^{2p+1}\int_e|\{\nabla_h u\}\cdot n_e|^2 ds\right)^{1/2}\\ &\le C\,h^p\|v\|_s\|u\|_{2,h}, \end{aligned}\tag{15.46}$$

where $\|u\|_{2,h}^2 = \sum_K\|u\|_{2,K}^2$. Note that the bilinear form remains stable with respect to the norm in (15.45) if the lower bound for η_e is large enough.

For the method of Brezzi et al. ($j = 7$), take the penalty term as

$$\alpha^r(u,v) = \sum_{e\in\mathcal{E}_h}\int_e h_e^{-2p} r_e([u])\cdot r_e([v])\,ds.$$

Again, we need to define a new norm through the relation

$$\|v\|_{ss}^2 = |v|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2|v|_{2,K}^2 + \alpha^r(v,v). \tag{15.47}$$

Boundedness and stability of $B_h^{(7)}$ hold, with respect to the norm $\|\cdot\|_{ss}$. We also have

$$\sum_{e\in\mathcal{E}_h}\int_e\{\nabla_h u\}\cdot[v]\,ds \le C\,h^p\|v\|_{ss}\|u\|_{2,h}\quad \forall\, u,v\in V(h). \tag{15.48}$$

Through arguments similar to that used in Theorem 118, using (15.46), and (15.48), we obtain the following theorems.

Theorem 111. *Let u and u_h be the solutions of* (15.1) *and* (15.114) *respectively. Assume $u\in H^{p+1}(\Omega)$, then, if the lower bound for the η_e is large enough, for the BabuVska-Zlámal DG method* ($j = 6$), *we have*

$$\|u - u_h\|_s \le Ch^p,$$

and for the method of Brezzi et al. ($j = 7$),

$$\|u - u_h\|_{ss} \le Ch^p,$$

where C is a positive constant that depends on $\|u\|_{p+1}$, the angle condition, the polynomial degree, and a bound on the edge-dependent penalty parameter η.

We will not cover the error analysis for the unstable DG methods (Baumann-Oden and Bassi-Rebay DG schemes) in this note. We make a remark here, even these two methods are not stable in the strong sense, but they are weak stable, which also can make them have certain convergence order. In the following table, we make a summary about these DG methods.

Table Properties of DG methods for Poisson problem

methods	Consistency	Stability	Type	Condition	convergence
1. IP	√	√	α^j	$\eta_0 \geq \eta_*$	h^p
2. NIPG	√	√	α^j	$\eta_0 \geq 0$	h^p
3. LDG	√	√	α^j	$\eta_0 \geq 0$	h^p
4. Brezzi et al.	√	√	α^r	$\eta_0 \geq 0$	h^p
5. Bassi et al.	√	√	α^r	$\eta_0 \geq 3$	h^p
6. BabuVska-Zlámal	×	√	α^j	$\eta_0 \approx h^{-2p}$	h^p
7. Brezzi et al.(Incon)	×	√	α^r	$\eta_0 \approx h^{-2p}$	h^p
8. Baumann-Oden	√	×	-	-	$h^p,\ p \geq 2$
9. Bassi-Rebay	√	×	-	-	$[h^p]$

Compared to other numerical methods, DG methods have many good features which make them are popular today, but the biggest disadvantage is that DG methods ”double” the freedom. Some researches are still on going in this direction of reducing this influence while keep the original advantages.

15.6 Hybridized discontinuous Galerkin methods

The DG methods are attracting the interest of many scientists because they discretize the equations in an element-by-element fashion through a Galerkin formulation which can give rise to locally conservative methods. They can handle any type of mesh, element shape and basis functions: They are ideally suited for hp-adaptivity. They have a built-in stabilization mechanism which does not degrade their (high-order) accuracy. They can be applied to a wide variety of partial differential equations.

However, the DG methods (for second-order elliptic equations) have been criticized for some reasons. For the same mesh and the same polynomial degree, the number of globally coupled degrees of freedom of the DG methods is much bigger than those of the CG method. Moreover, the orders of convergence of both the vector and scalar variables are also the same. For the same mesh and the same index, the number of globally coupled degrees if freedom of the DG methods are much bigger than those of the hybridized version of the RT and BDM methods. Moreover, the orders of convergence of both the vector and the local average of the scalar variables are smaller by one.

The HDG methods are obtained by discretizing characterizations of the exact solution written in terms of many local problems, one for each element of the mesh $\mathcal{T}_h$, with suitably chosen data, and in terms of a single global problem that actually determines them. This permits an efficiently implementation since they inherit the above-mentioned structure of the exact solution. This is what renders them efficiently implementable, especially within the framework of hp-adaptive methods, as is typical of DG methods.

The way in which they are defined allows them to be, in some instances, more accurate than already existing DG methods. In fact, in some cases when standard DG methods do not converge, HDG methods do. The HDG methods can be used for steady-state problems and for time-dependent problems when implicit time-marching methods are used. However, they might also be defined for explicit time-marching schemes.

15.6.1 Derivation of HDG

Following is the guidelines for devising the HDG methods.

1. Use a characterization of the exact solution in terms of solutions of local problems and transmission conditions.
2. Use discontinuous approximations for both the solution inside each element and its trace on the element boundary.
3. Define the local solvers by using a Galerkin method to weakly enforce the equations on each element.

4. Define a global problem by weakly imposing the transmission conditions.

Consider the model problem:

$$\begin{cases} -\nabla\cdot(\alpha(x)\nabla u) = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

Let us use this model problem as an example to show how to derive HDG method. First, we rewrite it as

$$c\mathbf{p} + \nabla u = 0, \quad \text{in } \Omega \tag{15.49}$$

$$\nabla\cdot\mathbf{p} = f \quad \text{in } \Omega, \tag{15.50}$$

$$u = g \quad \text{on } \partial\Omega. \tag{15.51}$$

Here, $c = \alpha^{-1}$ is a matrix-valued function which is symmetric and uniformly positive definite on Ω. We have that the exact solution satisfies the local problems

$$c\mathbf{p} + \nabla u = 0, \quad \text{in } K \tag{15.52}$$

$$\nabla\cdot\mathbf{p} = f \quad \text{in } K, \tag{15.53}$$

and it is well-defined with a boundary condition.

$$u = \widehat{u} \quad \text{on } \partial K. \tag{15.54}$$

So we can obtain $(\mathbf{p}, u)$ in K in terms of f and $\widehat{u}$ on ∂K by solving (15.52), (15.53) and (15.54). On the element $K \in \mathcal{T}_h$, given $\widehat{u}$ on ∂K and f , we have that $(\mathbf{p}, u)$ satisfies the equations

$$(c\mathbf{p}, \mathbf{q})_K - (u, \nabla\cdot\mathbf{q})_K + \langle\widehat{u}, \mathbf{q}\cdot\mathbf{n}\rangle_{\partial K} = 0, \tag{15.55}$$

$$-(\mathbf{p}, \nabla v)_K + \langle\widehat{\mathbf{p}}\cdot\mathbf{n}, v\rangle_{\partial K} = (f, v)_K, \tag{15.56}$$

for all $(q, v) \in \mathbf{Q}(K) \times V(K)$, where

$$\widehat{\mathbf{p}}\cdot\mathbf{n} = \mathbf{p}\cdot\mathbf{n} \quad \text{on } \partial K. \tag{15.57}$$

In discretization, we solve $(\mathbf{p}_h, u_h) \in \mathbf{Q}_h \times V_h$ (two DG spaces) in terms of $(\widehat{u}_h, f)$ by

$$(c\mathbf{p}_h, \mathbf{q}_h)_K - (u_h, \nabla\cdot\mathbf{q}_h)_K + \langle\widehat{u}_h, \mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K} = 0, \tag{15.58}$$

$$-(\mathbf{p}_h, \nabla v_h)_K + \langle\widehat{\mathbf{p}}_h\cdot\mathbf{n}, v_h\rangle_{\partial K} = (f, v_h)_K, \tag{15.59}$$

for all $(\mathbf{q}_h, v_h) \in \mathbf{Q}_h(K) \times V_h(K)$, where we define

$$\widehat{\mathbf{p}}_h\cdot\mathbf{n} = \mathbf{p}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h) \quad \text{on } \partial K. \tag{15.60}$$

We define the numerical flux $\widehat{\mathbf{p}}_h\cdot\mathbf{n}$ in this way because we want that the numerical trace $\widehat{\mathbf{p}}_h\cdot\mathbf{n}$ only depends on $\mathbf{p}_h\cdot\mathbf{n}$, u_h and $\widehat{u}_h$, and the dependence is linear. In addition, we want the numerical trace $\widehat{\mathbf{p}}_h\cdot\mathbf{n}$ is consistent, that is, $\widehat{\mathbf{p}}_h\cdot\mathbf{n} = \mathbf{p}_h\cdot\mathbf{n}$ if $u_h = \widehat{u}_h$.

Theorem 112. *The local solver on K is well defined if, for each $K \in \mathcal{T}_h$, one of the following condition satisfies*

1. $\nabla\cdot\mathbf{Q}_h(K) \subset \nabla V_h(K)$, *and* $\tau \geq 0$ *on* ∂K.
2. $\nabla V_h(K) \subset \mathbf{Q}(K)$, *and* $\tau > 0$ *on* ∂K.

Proof. The system is square, so we only need to prove that $\mathbf{p}_h = 0$ and $u_h = 0$ if $\widehat{u}_h = 0$ and $f = 0$. Let $(\mathbf{q}_h, v_h) = (\mathbf{p}_h, u_h)$ in (15.58) and (15.59), then

$$\begin{aligned}(c\mathbf{p}_h, \mathbf{p}_h)_K - (u_h, \nabla \cdot \mathbf{p}_h)_K &= 0,\\ -(\mathbf{p}_h, \nabla u_h)_K + \langle \widehat{p}_h \cdot \mathbf{n}, u_h\rangle_{\partial K} &= 0.\end{aligned}$$

So

$$(c\mathbf{p}_h, \mathbf{p}_h)_K + \langle (\widehat{\mathbf{p}}_h - \mathbf{p}_h) \cdot \mathbf{n}, u_h\rangle_{\partial K} = 0,$$

and note that $\widehat{\mathbf{p}}_h \cdot \mathbf{n} = \mathbf{p}_h \cdot \mathbf{n} + \tau u_h$, we get

$$(c\mathbf{p}_h, \mathbf{p}_h)_K + \langle \tau u_h, u_h\rangle_{\partial K} = 0,$$

which implies that $\mathbf{p}_h = 0$ on K and $u_h = 0$ on ∂K. Then from (15.58), we obtain

$$-(u_h, \nabla \cdot \mathbf{q}_h)_K = 0$$

for all $\mathbf{q}_h \in \mathbf{Q}(K)$. Hence, if $\nabla \cdot \mathbf{Q}_h(K) \subset \nabla V_h(K)$, we have $u_h = 0$ on K. Or, do integration by part, we have

$$(\nabla u_h, \mathbf{q}_h)_K = 0,$$

then $\nabla V_h(K) \subset \mathbf{Q}(K)$ implies $\nabla u_h = 0$, which implies that $u_h = 0$. This finishes the proof. □

There are three key ideas for HDG.

1. Introduce appropriate approximation of u_h on ∂K, i.e., we have a new additional function $\widehat{u}_h \in M_h(\mathcal{E}_h)$.
2. Define $\widehat{\mathbf{p}}_h \cdot \mathbf{n} = \mathbf{p}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$ on ∂K.
3. If $\widehat{u}_h$ is known, then u_h and $\mathbf{p}_h$ can be solved locally, i.e., we can eliminate the local D.O.F of u_h and $\mathbf{p}_h$, and get a global system only for $\widehat{u}_h$. To do this, for each face $e \in \mathcal{E}_h^0$, we take $\widehat{u}_h \in M_h(e)$, and determine $\widehat{u}_h$ by requiring that,

$$\langle [\widehat{\mathbf{p}}_h], \widehat{v}_h\rangle_e = 0, \quad \forall \widehat{v}_h \in M_h(e) \quad \text{if } e \in \mathcal{E}_h^0 \tag{15.61}$$

$$\widehat{u} = g \quad \text{if } e \in \mathcal{E}_h^{\partial}, \tag{15.62}$$

Theorem 113. *The numerical trace $\widehat{u}_h$ is well defined if, for each $K \in \mathcal{T}_h$, one of the following condition satisfies*

1. $\nabla \cdot \mathbf{Q}_h(K) \subset \nabla V_h(K)$, *and* $\tau \geq 0$ *on* ∂K.
2. $\nabla V_h(K) \subset \mathbf{Q}(K)$, *and* $\tau > 0$ *on* ∂K.

Proof. The system is square. Set $u_D = 0$, $f = 0$, and let $\widehat{v}_h = \widehat{u}_h$ in (15.121) and (15.122), we have

$$\begin{aligned}0 &= \sum_{e\in\mathcal{E}_h^0} \langle \widehat{u}_h, [\widehat{\mathbf{p}}_h]\rangle_e = \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h, \widehat{\mathbf{p}}_h \cdot \mathbf{n}\rangle_{\partial K}\\ &= \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h, \mathbf{p}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)\rangle_{\partial K}\\ &= \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h, \mathbf{p}_h \cdot \mathbf{n}\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h - u_h, \tau(u_h - \widehat{u}_h)\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h} \langle u_h, \tau(u_h - \widehat{u}_h)\rangle_{\partial K}\\ &= \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h, \mathbf{p}_h \cdot \mathbf{n}\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h - u_h, \tau(u_h - \widehat{u}_h)\rangle_{\partial K}\\ &\quad + \sum_{K\in\mathcal{T}_h} \langle u_h, \widehat{\mathbf{p}}_h \cdot \mathbf{n}\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h} \langle u_h, \mathbf{p}_h \cdot \mathbf{n}\rangle_{\partial K}\end{aligned}$$

Recall the local problem for $(\mathbf{q}_h, v_h) = (\mathbf{p}_h, u_h)$, we have

$$\begin{aligned}(c\mathbf{p}_h, \mathbf{p}_h)_K - (u_h, \nabla\cdot\mathbf{p}_h)_K + \langle\widehat{u}_h, \mathbf{p}_h\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{p}_h, \nabla u_h)_K + \langle\widehat{\mathbf{p}}_h\cdot\mathbf{n}, u_h\rangle_{\partial K} &= 0.\end{aligned}$$

Then

$$0 = -\sum_{K\in\mathcal{T}_h}\langle\widehat{u}_h, \widehat{\mathbf{p}}_h\cdot\mathbf{n}\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h}(c\mathbf{p}_h, \mathbf{p}_h)_K + \sum_{K\in\mathcal{T}_h}\langle u_h - \widehat{u}_h, \tau(u_h - \widehat{u}_h)\rangle_{\partial K}$$

which means $\mathbf{p}_h = 0$ on each $K\in\mathcal{T}_h$ and $u_h = \widehat{u}_h$ on each edge $e\in\mathcal{E}_h$. Then from (15.58), we obtain

$$-(u_h, \nabla\cdot\mathbf{q}_h)_K = 0$$

for all $\mathbf{q}_h\in\mathbf{Q}(K)$. Hence, if $\nabla\cdot\mathbf{Q}_h(K)\subset\nabla V_h(K)$, we have $u_h = 0$ on K. Or, do integration by part, we have

$$(\nabla u_h, \mathbf{q}_h)_K = 0,$$

then $\nabla V_h(K)\subset\mathbf{Q}(K)$ implies $\nabla u_h = 0$, which means that u_h is a constant. Then we finish the proof by applying $u_h = \widehat{u}_h = 0$ on $\partial\Omega$. □

The system can be given by

$$\sum_{K\in\mathcal{T}_h}(c\mathbf{p}_h, \mathbf{q}_h)_K - \sum_{K\in\mathcal{T}_h}(u_h, \nabla\cdot\mathbf{q}_h)_K + \sum_{K\in\mathcal{T}_h}\langle\widehat{u}_h, \mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K} = 0, \tag{15.63}$$

$$-\sum_{K\in\mathcal{T}_h}(\mathbf{p}_h, \nabla v_h)_K + \sum_{K\in\mathcal{T}_h}\langle\widehat{\mathbf{p}}_h\cdot\mathbf{n}, v_h\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h}(f, v_h)_K, \tag{15.64}$$

$$\sum_{K\in\mathcal{T}_h}\langle[\widehat{\mathbf{p}}_h], \widehat{v}_h\rangle_{\partial K} = 0 \tag{15.65}$$

Take $\widehat{\mathbf{p}}_h\cdot\mathbf{n} = \mathbf{p}_h\cdot\mathbf{n} + \tau(u_h - \widehat{u}_h)$ into it, and by some manipulation, we have

$$\sum_{K\in\mathcal{T}_h}(c\mathbf{p}_h, \mathbf{q}_h)_K - \sum_{K\in\mathcal{T}_h}(u_h, \nabla\cdot\mathbf{q}_h)_K + \sum_{K\in\mathcal{T}_h}\langle\widehat{u}_h, \mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K} = 0, \tag{15.66}$$

$$-\sum_{K\in\mathcal{T}_h}(\nabla\cdot\mathbf{p}_h, v_h)_K - \sum_{K\in\mathcal{T}_h}\tau\langle u_h, v_h\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h}\tau\langle\widehat{u}_h, v_h\rangle_{\partial K} = \sum_{K\in\mathcal{T}_h}(-f, v_h)_K, \tag{15.67}$$

$$\sum_{K\in\mathcal{T}_h}\langle\mathbf{p}_h\cdot\mathbf{n}, \widehat{v}_h\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h}\tau\langle u_h, \widehat{v}_h\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\tau\langle\widehat{u}_h, \widehat{v}_h\rangle_{\partial K} = 0. \tag{15.68}$$

15.6.2 Comparison between HDG and WG

In this subsection, we compare HDG and WG. Consider the following problem

$$\mathbf{p} + \alpha\nabla u = 0, \quad \text{in } \Omega \tag{15.69}$$

$$\nabla\cdot\mathbf{p} = f \quad \text{in } \Omega, \tag{15.70}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{15.71}$$

Then the HDG can be written as

$$\sum_{K\in\mathcal{T}_h}(\mathbf{p}_h,\mathbf{q}_h)_K-\sum_{K\in\mathcal{T}_h}(u_h,\nabla\cdot(\alpha\mathbf{q}_h))_K+\sum_{K\in\mathcal{T}_h}\langle\widehat{u}_h,\alpha\mathbf{q}_h\cdot\mathbf{n}\rangle_{\partial K}=0,\tag{15.72}$$

$$\sum_{K\in\mathcal{T}_h}(\nabla\cdot\mathbf{p}_h,v_h)_K+\sum_{K\in\mathcal{T}_h}\tau\langle u_h,v_h\rangle_{\partial K}-\sum_{K\in\mathcal{T}_h}\tau\langle\widehat{u}_h,v_h\rangle_{\partial K}=\sum_{K\in\mathcal{T}_h}(f,v_h)_K,\tag{15.73}$$

$$\sum_{K\in\mathcal{T}_h}\langle\mathbf{p}_h\cdot\mathbf{n},\widehat{v}_h\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\tau\langle u_h,\widehat{v}_h\rangle_{\partial K}-\sum_{K\in\mathcal{T}_h}\tau\langle\widehat{u}_h,\widehat{v}_h\rangle_{\partial K}=0.\tag{15.74}$$

First, we rewrite (15.73) as

$$-\sum_{K\in\mathcal{T}_h}(\mathbf{p}_h,\nabla_w v_h)_K+\sum_{K\in\mathcal{T}_h}\langle\mathbf{p}_h\cdot\mathbf{n},\widehat{v}_h\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\langle\tau(u_h-\widehat{u}_h),v_h\rangle_{\partial K}=\sum_{K\in\mathcal{T}_h}(f,v_h)_K,\tag{15.75}$$

by the definition of weak gradient

$$\langle\nabla_w u,\mathbf{q}\rangle_K=-(u,\nabla\cdot\mathbf{q})_K+\langle\widehat{u},\mathbf{q}\cdot\boldsymbol{n}\rangle_{\partial K}.$$

Similarly, we can rewrite (15.72) as

$$\sum_{K\in\mathcal{T}_h}(\mathbf{p}_h,\mathbf{q}_h)_K+\sum_{K\in\mathcal{T}_h}(\nabla_w^{\alpha}u_h,\alpha\mathbf{q}_h)_K=0,$$

where

$$\langle\nabla_w^{\alpha}u,\alpha\mathbf{q}\rangle_K=-(u,\nabla\cdot(\alpha\mathbf{q}))_K+\langle\widehat{u},\alpha\mathbf{q}\cdot\boldsymbol{n}\rangle_{\partial K}.$$

Let $\mathbf{q}_h=\nabla_w v_h$ in above equation, we get

$$\sum_{K\in\mathcal{T}_h}(\mathbf{p}_h,\nabla_w v_h)_K+\sum_{K\in\mathcal{T}_h}(\nabla_w^{\alpha}u_h,\alpha\nabla_w v_h)_K=0,\tag{15.76}$$

Combining (15.75) and (15.76), we have

$$\sum_{K\in\mathcal{T}_h}(\nabla_w^{\alpha}u_h,\alpha\nabla_w v_h)_K+\sum_{K\in\mathcal{T}_h}\langle\mathbf{p}_h\cdot\mathbf{n},\widehat{v}_h\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\langle\tau(u_h-\widehat{u}_h),v_h\rangle_{\partial K}=\sum_{K\in\mathcal{T}_h}(f,v_h)_K,\tag{15.77}$$

Considering (15.74), (15.77) can be rewritten as

$$\sum_{K\in\mathcal{T}_h}(\nabla_w^{\alpha}u_h,\alpha\nabla_w v_h)_K-\sum_{K\in\mathcal{T}_h}\langle\tau(u_h-\widehat{u}_h),\widehat{v}_h\rangle_{\partial K}+\sum_{K\in\mathcal{T}_h}\langle\tau(u_h-\widehat{u}_h),v_h\rangle_{\partial K}=\sum_{K\in\mathcal{T}_h}(f,v_h)_K,\tag{15.78}$$

that is

$$\sum_{K\in\mathcal{T}_h}(\nabla_w^{\alpha}u_h,\alpha\nabla_w v_h)_K+\sum_{K\in\mathcal{T}_h}\langle\tau(u_h-\widehat{u}_h),v_h-\widehat{v}_h\rangle_{\partial K}=\sum_{K\in\mathcal{T}_h}(f,v_h)_K.\tag{15.79}$$

Therefore, we can see that the only difference between HDG and WG is ∇_w^{α}, if α is a constant matrix, the HDG is the hybrid method for WG. If α is not constant, then the two methods are not equivalent, but share the similar ideas for adopting the weak derivative.

15.6.3 Bilinear Form for Lagrange multiplier

Let us slip (15.52), (15.53) and (15.54) as two local problems as follow:

$$
\begin{aligned}
c\mathbf{Q}_{\widehat{u}} + \nabla U_{\widehat{u}} &= 0 \quad \text{in } K, && (15.80)\\
\nabla\cdot\mathbf{Q}_{\widehat{u}} &= 0 \quad \text{in } K, && (15.81)\\
U_{\widehat{u}} &= \widehat{u} \quad \text{on } \partial K, && (15.82)
\end{aligned}
$$

and

$$
\begin{aligned}
c\mathbf{Q}_{f} + \nabla U_{f} &= 0 \quad \text{in } K, && (15.83)\\
\nabla\cdot\mathbf{Q}_{f} &= f \quad \text{in } K, && (15.84)\\
U_{f} &= 0 \quad \text{on } \partial K. && (15.85)
\end{aligned}
$$

Furthermore, the function $\widehat{u}$ can be determined by

$$
\begin{aligned}
-[\widehat{\mathbf{Q}}_{\widehat{u}}] &= [\widehat{\mathbf{Q}}_{f}], \quad \text{if } e\in\mathcal{E}_h^0 && (15.86)\\
\widehat{u} &= u_D \quad \text{if } e\in\mathcal{E}_h^{\partial}, && (15.87)
\end{aligned}
$$

Then we have

$$
(\mathbf{p}_h, u_h) = (\mathbf{Q}_{\widehat{u}}, U_{\widehat{u}}) + (\mathbf{Q}_f, U_f).
$$

The numerical trace $\widehat{u}_h$ is the element of the space

$$
M_h(u_D) := \{\widehat{v}_h \in L^2(\mathcal{E}_h) : \widehat{v}_h|_e \in M(e)\ \forall e\in\mathcal{E}_h, \widehat{v}_h|_{\partial\Omega} := u_D\},
$$

and satisfies

$$
a_h(\widehat{u}_h,\mu) = l_h(\mu) \quad \forall \mu\in M_h(0) \tag{15.88}
$$

where $a_h(\mu,\lambda) := -\sum_{K\in\mathcal{T}_h}\langle \mu, \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n}\rangle_{\partial K}$, and $l_h(\mu) = \sum_{K\in\mathcal{T}_h}\langle \mu, \widehat{\mathbf{Q}}_f\cdot\mathbf{n}\rangle_{\partial K}$.

Theorem 114.

$$
a_h(\mu,\lambda) := \sum_{K\in\mathcal{T}_h}(c\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K + \sum_{K\in\mathcal{T}_h}\langle \tau(U_\mu-\mu), (U_\lambda-\lambda)\rangle_{\partial K}
$$

is positive definite on $M_h(0)\times M_h(0)$.

Proof.

$$
\begin{aligned}
a_h(\mu,\lambda) &= -\sum_{K\in\mathcal{T}_h}\langle \mu, \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n}\rangle_{\partial K}\\
&= -\sum_{K\in\mathcal{T}_h}\langle \mu, \mathbf{Q}_\lambda\cdot\mathbf{n} + \tau(U_\lambda-\lambda)\rangle_{\partial K}\\
&= -\sum_{K\in\mathcal{T}_h}\langle \mu, \mathbf{Q}_\lambda\cdot\mathbf{n}\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\langle \mu-U_\mu, \tau(U_\lambda-\lambda)\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\langle U_\mu, \tau(U_\lambda-\lambda)\rangle_{\partial K}\\
&= -\sum_{K\in\mathcal{T}_h}\langle \mu, \mathbf{Q}_\lambda\cdot\mathbf{n}\rangle_{\partial K} - \sum_{K\in\mathcal{T}_h}\langle \mu-U_\mu, \tau(U_\lambda-\lambda)\rangle_{\partial K}\\
&\quad - \sum_{K\in\mathcal{T}_h}\langle U_\mu, \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n})\rangle_{\partial K} + \sum_{K\in\mathcal{T}_h}\langle U_\mu, \mathbf{Q}_\lambda\cdot\mathbf{n})\rangle_{\partial K}
\end{aligned}
$$

Note that the weak form of the local problem (15.83), (15.85) with $\widehat{u}_h = \mu$, and (15.84) with $\widehat{u}_h = \lambda$ is

$$\begin{aligned}(c\mathbf{Q}_\mu, \mathbf{w})_K - (U_\mu, \nabla\cdot\mathbf{w})_K + \langle \mu, \mathbf{w}\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{Q}_\lambda, \nabla v)_K + \langle \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n}, v\rangle_{\partial K} &= 0,\end{aligned}$$

for all $(\mathbf{w}, v) \in \mathbf{W}(K)\times V(K)$. Let $(\mathbf{w}, v) = (\mathbf{Q}_\lambda, U_\mu)$ in the above equation, we get

$$\begin{aligned}(c\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K - (U_\mu, \nabla\cdot\mathbf{Q}_\lambda)_K + \langle \mu, \mathbf{Q}_\lambda\cdot\mathbf{n}\rangle_{\partial K} &= 0,\\ -(\mathbf{Q}_\lambda, \nabla U_\mu)_K + \langle \widehat{\mathbf{Q}}_\lambda\cdot\mathbf{n}, U_\mu\rangle_{\partial K} &= 0.\end{aligned}$$

So
$$a_h(\mu,\lambda) := \sum_{K\in\mathcal{T}_h}(c\,\mathbf{Q}_\mu, \mathbf{Q}_\lambda)_K + \sum_{K\in\mathcal{T}_h}\langle \tau(U_\mu - \mu), (U_\lambda - \lambda)\rangle_{\partial K}$$

which completes the proof. □

15.6.4 Equivalence between lowest-order RT mixed method and CR nonconforming method

Consider the Dirichlet boundary value problem

$$\begin{aligned}-\nabla\cdot(A\nabla u) &= f \quad \text{in } \Omega,\\ u &= 0 \quad \text{on } \partial\Omega,\end{aligned} \tag{15.89}$$

where A is positive-definite matrix.

It can be discretized in the mixed formulation: finding $(p, u) \in RT_0^0(\mathcal{T}_h)\times V_0^{-1}(\mathcal{T}_h)$ such that

$$\begin{aligned}(Cp, q)_\Omega - (\mathrm{div} q, u)_\Omega &= 0,\\ -(\mathrm{div} p, v)_\Omega &= -(f, v)_\Omega,\end{aligned} \tag{15.90}$$

for any $(q, v) \in RT_0^0(\mathcal{T}_h)\times V_0^{-1}$. Here $C := A^{-1}$, and $RT_0^0(\mathcal{T}_h)$ is the lowest-order Raviart-Thomas elements and

$$V_0^{-1}(\mathcal{T}_h) = \{v\in L^2(\Omega) \mid v|_T \in \mathcal{P}_0(T)\ \forall T\in\mathcal{T}_h\}.$$

By hybridization, the mixed method is equivalent to the hybridized mixed method: finding $(p, u, \lambda) \in RT_0^0(\mathcal{T}_h)\times V_0^{-1}(\mathcal{T}_h)\times M_{-1}^0(\mathcal{E}_h)$ such that

$$\begin{aligned}\sum_{T\in\mathcal{T}_h}(Cp, q)_T - \sum_{T\in\mathcal{T}_h}(\mathrm{div} q, u)_T + \sum_{T\in\mathcal{T}_h}\langle \lambda, q\cdot v\rangle_{\partial T} &= 0,\\ -\sum_{T\in\mathcal{T}_h}(\mathrm{div} p, v)_T &= -(f, v)_\Omega,\\ \sum_{T\in\mathcal{T}_h}\langle \mu, p\cdot v\rangle_{\partial T} &= 0\end{aligned} \tag{15.91}$$

for any $(q, v, \mu) \in RT_0^0(\mathcal{T}_h)\times V_0^{-1}(\mathcal{T}_h)\times M_{-1}^0(\mathcal{E}_h)$. Here $RT_{-1}^0(\mathcal{T}_h)$ is the lowest-order Raviart-Thomas elements without interelement continuity and

$$M_0^{-1}(\mathcal{E}_h) = \{\mu\in L^2(\mathcal{E}_h) \mid \mu|_e \in \mathcal{P}_0(e)\ \forall e\in\mathcal{E}_h^0, \mu|_{\partial\Omega} = 0\}.$$

We define the following lifting operators

$$\mathcal{Q} : M^0_{-1}(\mathcal{E}_h) \mapsto RT^0_0(\mathcal{T}_h),$$

$$\mathcal{U} : M^0_{-1}(\mathcal{E}_h) \mapsto V^{-1}_0(\mathcal{T}_h),$$

$$\bar{\mathcal{Q}} : L^2(\Omega) \mapsto RT^0_0(\mathcal{T}_h),$$

$$\bar{\mathcal{U}} : L^2(\Omega) \mapsto V^{-1}_0(\mathcal{T}_h),$$

satisfying following equations

$$\begin{aligned} (C\mathcal{Q}\lambda, q)_T - (\mathrm{div} q, \mathcal{U}\lambda)_T &= \langle \lambda, q\cdot\nu\rangle_{\partial T}, \\ -(\mathrm{div}\mathcal{Q}\lambda, v)_T &= 0, \end{aligned} \tag{15.92}$$

and

$$\begin{aligned} (C\bar{\mathcal{Q}}f, q)_T - (\mathrm{div} q, \bar{\mathcal{U}}f)_T &= 0, \\ -(\mathrm{div}\bar{\mathcal{Q}}f, v)_T &= -(f, v)_T, \end{aligned} \tag{15.93}$$

for any $q \in RT_0(T)$, $v \in V_0(T)$ and $T \in \mathcal{T}_h$.

We have the following theorem

Theorem 115. *The solution of the mixed method* (15.90) *satisfies*

$$p = \mathcal{Q}\lambda + \bar{\mathcal{Q}}f, \qquad u = \mathcal{U}\lambda + \bar{\mathcal{U}}f. \tag{15.94}$$

And the Lagrangian multiplier λ is determined by following equation

$$(C\mathcal{Q}\lambda, \mathcal{Q}\mu)_\Omega = (f, \mathcal{U}\mu)_\Omega \qquad \forall \mu \in M^{-1}_0(\mathcal{T}_h). \tag{15.95}$$

Denote by $CR(\mathcal{T}_h)$ the Crouzeix-Raviart nonconforming element. Let $\mathcal{S} : CR(\mathcal{T}_h) \mapsto M^{-1}_0(\mathcal{E}_h)$ be defined by

$$\mathcal{S}\psi = \mu,$$

where

$$\mu|e = \psi(m_e).$$

Here m_e is the midpoint of edge e. $\mathcal{S}$ is clearly an isomorphism.

Let P_C be the orthogonal projection from $(L^2(\Omega))^n$ onto $(V^{-1}_0(\mathcal{T}_h))^n$ with respect to the inner product

$$[p, q] = (Cp, q)_\Omega.$$

For any $\phi \in CR(\mathcal{T}_h)$ and $p_0 \in (V^{-1}_0(\mathcal{T}_h))^n$, we have

$$\begin{aligned} [A\nabla_h\phi, p_0] &= (\nabla_h\phi, p_0)_\Omega \\ &= \sum_{T\in\mathcal{T}_h} \langle \phi, p_0\cdot\nu\rangle_{\partial T} \\ &= \sum_{T\in\mathcal{T}_h} \langle \mathcal{S}\phi, p_0\cdot\nu\rangle_{\partial T} \\ &= [\mathcal{Q}\mathcal{S}\phi, p_0]. \end{aligned} \tag{15.96}$$

Since $\mathcal{Q}\mathcal{S}\phi$ is divergence-free, by the property of lowest-order RT element, we have $\mathcal{Q}\mathcal{S}\phi \in (V^{-1}_0(\mathcal{T}_h))^n$. Thus

$$\mathcal{Q}\mathcal{S}\phi = P_C A\nabla_h\phi$$

Let $\phi \in CR(\mathcal{T}_h)$ be defined by

$$\mathcal{S}\phi = \lambda$$

where λ is the solution of the system (15.95) of Lagrangian multiplier. Then,

$$\begin{aligned}[P_C A\nabla_h\phi, P_C A\nabla_h\psi] &= (C\mathcal{Q}\mathcal{S}\phi, \mathcal{Q}\mathcal{S}\psi)_\Omega \\ &= (f, \mathcal{U}\mathcal{S}\psi)_\Omega \\ &= (f, P_0\psi)_\Omega.\end{aligned} \tag{15.97}$$

Here P_0 is the L^2 projection onto $V_0^{-1}(\mathcal{T})$. The last equality follows from the definition of lifting operators. Then we have following theorem

Theorem 116. *Suppose $\phi \in CR(\mathcal{T}_h)$ be the solution of following equation*

$$(CP_C A\nabla_h\phi, P_C A\nabla_h\psi)_\Omega = (f, P_0\psi)_\Omega \qquad \forall\psi \in CR(\mathcal{T}_h), \tag{15.98}$$

and $\lambda \in M_0^{-1}(\mathcal{E}_h)$ be the solution of the multiplier system (15.95). *We have*

$$\mathcal{S}\phi = \lambda.$$

If A is the constant matrix, the CR system can be written as

$$(A\nabla_h\phi, \nabla_h\psi)_\Omega = (f, P_0\psi)_\Omega \qquad \forall\psi \in CR(\mathcal{T}_h).$$

15.7 Unified analysis for DG methods

In this section, we try to use a unified way to rewrite and analyze DG methods. Let us still consider the Poisson problem (15.1). To do it, we first rewrite the Poisson problem as the first order system

$$\mathbf{p} = \nabla u, \quad \text{in } \Omega \tag{15.99}$$

$$-\nabla\cdot\mathbf{p} = f \quad \text{in } \Omega, \tag{15.100}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{15.101}$$

We multiply the equations (15.99) and (15.100) by test functions $\mathbf{w}$ and v, respectively, and integrate on a subset $K \subset \Omega$. We get by integration by part,

$$\int_K \mathbf{p}\cdot\mathbf{w}\,dx = -\int_K u\,\nabla\cdot\mathbf{w}\,dx + \int_{\partial K} u\,\mathbf{n}_K\cdot\mathbf{w}\,ds \tag{15.102}$$

$$\int_K \mathbf{p}\cdot\nabla v\,dx = \int_K f\,v\,dx + \int_{\partial K}\mathbf{p}\cdot\,n_K\,v\,ds, \tag{15.103}$$

where n_K is the outward normal unit vector to ∂K. In above equations, we append subscript h on $\mathbf{p}$, u, ∇ and v, and use numerical traces $\widehat{u_h}$ and $\widehat{\mathbf{p}_h}$ to approximate u and $\mathbf{p}$ over element edges to get

$$\begin{aligned}\int_K \mathbf{p}_h\cdot\mathbf{w}_h\,dx &= -\int_K u_h\,\nabla_h\cdot\mathbf{w}_h\,dx + \int_{\partial K}\widehat{u_h}\,n_K\cdot\mathbf{w}_h\,ds = \int_K \nabla_w u_h\cdot\mathbf{w}_h\,dx \\ \int_K \mathbf{p}_h\cdot\nabla_h\mathbf{v}_h\,dx &= \int_K f\,v_h\,dx + \int_{\partial K}\widehat{\mathbf{p}_h}\cdot\,n_K\,v_h\,ds.\end{aligned}$$

So we can see that $\mathbf{p}_h = \nabla_w u_h$ on each K, where ∇_w is the week gradient defined by

Definition 13. *For any $u \in W(K)$, the weak gradient of u is defined as a bounded linear functional $\nabla_w u$ in $H(div; K)$ whose action on each $q \in H(div; K)$ is given by*

$$\langle\nabla_w u, q\rangle_K = -(u, \nabla\cdot q)_K + \langle\widehat{u}, q\cdot\boldsymbol{n}\rangle_{\partial K},$$

where $\boldsymbol{n}$ is the outward normal direction on ∂K.

15.7.1 DG formulation

In this section, we derive DG formulations.
We add over all the elements to obtain

$$\int_\Omega \mathbf{p}_h \cdot \mathbf{w}_h\, dx = -\int_\Omega u_h\, \nabla_h \cdot \mathbf{w}_h\, dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \widehat{u_h}\, n_K \cdot \mathbf{w}_h\, ds \qquad \forall\, \mathbf{w}_h \in W_h, \tag{15.104}$$

$$\int_\Omega \mathbf{p}_h \cdot \nabla_h v_h\, dx = \int_\Omega f\, v_h\, dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \widehat{\mathbf{p}_h} \cdot\, n_K\, v_h\, ds \qquad \forall\, v_h \in V_h, \tag{15.105}$$

for all $(\mathbf{w}_h, v_h) \in W_h \times V_h$ and all $K \in \mathcal{T}_h$. The numerical traces $\widehat{\mathbf{p}_h}$ and $\widehat{u_h}$ will be selected to guarantee consistency and stability of the above scheme. Different choices of the numerical fluxes leads to different DG methods.

By integration by part, we get

$$\int_\Omega \mathbf{p}_h \cdot \mathbf{w}_h\, dx = \int_\Omega \nabla_h u_h \cdot \mathbf{w}_h\, dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\widehat{u_h} - u_h)\, n_K \cdot \mathbf{w}_h\, ds \qquad \forall\, \mathbf{w}_h \in W_h, \tag{15.106}$$

$$\int_\Omega \mathbf{p}_h \cdot \nabla_h v_h\, dx = \int_\Omega f\, v_h\, dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \widehat{\mathbf{p}_h} \cdot\, n_K\, v_h\, ds \qquad \forall\, v_h \in V_h, \tag{15.107}$$

By the identity (23.62), the above equations can be written as

$$\int_\Omega \mathbf{p}_h \cdot \mathbf{w}_h\, dx = \int_\Omega \nabla_h u_h \cdot \mathbf{w}_h\, dx + \sum_{e\in\mathcal{E}_h} \int_e [\widehat{u_h} - u_h] \cdot \{\mathbf{w}_h\}\, ds + \sum_{e\in\mathcal{E}_h^i} \int_e \{\widehat{u_h} - u_h\}[\mathbf{w}_h]\, ds, \tag{15.108}$$

$$\int_\Omega \mathbf{p}_h \cdot \nabla_h v_h\, dx = \int_\Omega f\, v_h\, dx + \sum_{e\in\mathcal{E}_h} \int_e [v_h] \cdot \{\widehat{\mathbf{p}_h}\}\, ds + \sum_{e\in\mathcal{E}_h^i} \int_e \{v_h\}[\widehat{\mathbf{p}_h}]\, ds, \tag{15.109}$$

Let us choose

$$\begin{cases} \widehat{u_h} = \{u_h\} - \boldsymbol{\beta} \cdot [u_h] \quad \text{on } \mathcal{E}_h^i, \qquad \widehat{u_h} = 0 \quad \text{on } \partial\Omega, \\ \widehat{\mathbf{p}_h} = \alpha\{\nabla_w u_h\} + (1-\alpha)\{\nabla_h u_h\} + [\nabla_w u_h]\boldsymbol{\beta} - L([u_h]) \quad \text{on } \mathcal{E}_h^i, \\ \widehat{\mathbf{p}_h} = \alpha\{\nabla_w u_h\} + (1-\alpha)\{\nabla_h u_h\} - L([u_h]) \quad \text{on } \partial\Omega, \end{cases}$$

where $L([u_h]) = \frac{\eta}{h_e}[u_h]$ or $\eta_e r_e([u_h])$. Here $\boldsymbol{\beta} \in [L^2(\mathcal{E}^i)]^2$ is a vector-valued function which is constant on each edge. Note that $\nabla_w u_h = \mathbf{p}_h$. In such choice, we can see that if u_h is replaced by the exact solution u, $\widehat{u_h} = u$ and $\widehat{\mathbf{p}_h} = \mathbf{p}|_{\mathcal{E}_h}$ on $\mathcal{E}_h$, so the numerical fluxes are consistent and single valued, which means they are conservative.

Then we obtain

$$\int_\Omega \mathbf{p}_h \cdot \mathbf{w}_h\, dx = \int_\Omega \nabla_h u_h \cdot \mathbf{w}_h\, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\mathbf{w}_h\}\, ds - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\mathbf{w}_h]\, ds, \tag{15.110}$$

$$\begin{aligned} \int_\Omega \mathbf{p}_h \cdot \nabla_h v_h\, dx = \int_\Omega f\, v_h\, dx &+ \alpha \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_w u_h\} \cdot [v_h]\, ds + (1-\alpha) \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h u_h\} \cdot [v_h]\, ds \\ &+ \sum_{e\in\mathcal{E}_h^i} \int_e [\nabla_w u_h]\boldsymbol{\beta} \cdot [v_h]\, ds - \sum_{e\in\mathcal{E}_h} \int_e L([u_h]) \cdot [v_h]\, ds \end{aligned} \tag{15.111}$$

Choosing $\mathbf{w}_h = \nabla_h v_h$ in (15.110), we have

$$\int_\Omega \mathbf{p}_h \cdot \nabla_h v_h \, dx = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\nabla_h v_h\} \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\nabla_h v_h] \, ds. \tag{15.112}$$

Combining (15.111) and (15.112), we get

$$\begin{aligned}\int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\nabla_h v_h\} \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\nabla_h v_h] \, ds - \alpha \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_w u_h\} \cdot [v_h] \, ds \\ - (1-\alpha) \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h u_h\} \cdot [v_h] \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e [\nabla_w u_h] \boldsymbol{\beta} \cdot [v_h] \, ds + \sum_{e\in\mathcal{E}_h} \int_e L([u_h]) \cdot [v_h] \, ds = \int_\Omega f v_h \, dx.\end{aligned} \tag{15.113}$$

Then we obtain

$$B_h(u_h, v_h) = \int_\Omega f v_h \, dx. \tag{15.114}$$

where

$$\begin{aligned} B_h(u_h, v_h) = {} & \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\nabla_h v_h\} \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\nabla_h v_h] \, ds \\ & - \alpha \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_w u_h\} \cdot [v_h] \, ds - (1-\alpha) \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h u_h\} \cdot [v_h] \, ds \\ & - \sum_{e\in\mathcal{E}_h^i} \int_e [\nabla_w u_h] \boldsymbol{\beta} \cdot [v_h] \, ds + \sum_{e\in\mathcal{E}_h} \int_e L([u_h]) \cdot [v_h] \, ds. \end{aligned} \tag{15.115}$$

Recall (15.110), we have for any $\mathbf{w}_h \in W_h$

$$\int_\Omega \mathbf{p}_h \cdot \mathbf{w}_h \, dx = \int_\Omega \nabla_h u_h \cdot \mathbf{w}_h \, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\mathbf{w}_h\} \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\mathbf{w}_h] \, ds$$

Let us introduce the following lifting operator, $r : [L^2(\mathcal{E}_h)]^2 \to W_h$, , $r_e : [L^2(e)]^2 \to W_h$ and $l : L^2(\mathcal{E}_h^i) \to W_h$ are lifting operators defined by

$$\int_\Omega r(\mathbf{q}) \cdot \mathbf{w}_h \, dx = -\int_{\mathcal{E}_h} \mathbf{q} \cdot \{\mathbf{w}_h\} \, ds, \quad \forall\, \mathbf{w}_h \in W_h, \tag{15.116}$$

$$\int_\Omega r_e(\mathbf{q}) \cdot \mathbf{w}_h \, dx = -\int_e \mathbf{q} \cdot \{\mathbf{w}_h\} \, ds, \quad \forall\, \mathbf{w}_h \in W_h, \tag{15.117}$$

$$\int_\Omega l(v) \cdot \mathbf{w}_h \, dx = -\int_{\mathcal{E}_h^i} v[\mathbf{w}_h] \, ds \quad \forall\, \mathbf{w}_h \in W_h. \tag{15.118}$$

Then we have

$$\int_\Omega \mathbf{p}_h \cdot \mathbf{w}_h \, dx = \int_\Omega \nabla_h u_h \cdot \mathbf{w}_h \, dx + \int_\Omega r([u_h]) \cdot \mathbf{w}_h \, dx + \int_\Omega l(\boldsymbol{\beta} \cdot [u_h]) \cdot \mathbf{w}_h \, dx$$

Therefore,

$$\nabla_w u_h = \mathbf{p}_h = \nabla_h u_h + r([u_h]) + l(\boldsymbol{\beta} \cdot [u_h]),$$

Then the bilinear form can be rewritten as

(15.119)

$$\begin{aligned}
B_h(u_h, v_h) = & \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\nabla_h v_h\} \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\nabla_h v_h] \, ds \\
& - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h u_h\} \cdot [v_h] \, ds - \alpha \sum_{e\in\mathcal{E}_h} \int_e \{r([u_h]) + l(\boldsymbol{\beta} \cdot [u_h])\} \cdot [v_h] \, ds \\
& - \sum_{e\in\mathcal{E}_h^i} \int_e [\nabla_h u_h]\boldsymbol{\beta} \cdot [v_h] \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e [r([u_h]) + l(\boldsymbol{\beta} \cdot [u_h])]\boldsymbol{\beta} \cdot [v_h] \, ds + \sum_{e\in\mathcal{E}_h} \int_e L([u_h]) \cdot [v_h] \, ds \\
= & \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \sum_{e\in\mathcal{E}_h} \int_e [u_h] \cdot \{\nabla_h v_h\} \, ds - \sum_{e\in\mathcal{E}_h} \int_e \{\nabla_h u_h\} \cdot [v_h] \, ds \\
& - \sum_{e\in\mathcal{E}_h^i} \int_e \boldsymbol{\beta} \cdot [u_h][\nabla_h v_h] \, ds - \sum_{e\in\mathcal{E}_h^i} \int_e [\nabla_h u_h]\boldsymbol{\beta} \cdot [v_h] \, ds + \alpha \int_\Omega r([u_h]) + l(\boldsymbol{\beta} \cdot [u_h]) \cdot r([v_h]) \, dx \\
& + \int_\Omega (r([u_h]) + l(\boldsymbol{\beta} \cdot [u_h])) \cdot l(\boldsymbol{\beta} \cdot [v_h]) \, dx + \sum_{e\in\mathcal{E}_h} \int_e L([u_h]) \cdot [v_h] \, ds
\end{aligned}$$

We can see that the existing DG methods can be given by specific choices of $\alpha, \boldsymbol{\beta}$ and $L([u_h])$. For example,

1. if $\alpha = 0, \boldsymbol{\beta} = \mathbf{0}$ and $L([u_h]) = \frac{\eta}{h_e}[u_h]$, it is the IP method;
2. if $\alpha = 0, \boldsymbol{\beta} = \mathbf{0}$ and $L([u_h]) = \eta_e r_e([u_h])$, it is the method of Bassi et. al.;
3. if $\alpha = 1, \boldsymbol{\beta} \neq \mathbf{0}$ and $L([u_h]) = \frac{\eta}{h_e}[u_h]$, it is the LDG method;
4. if $\alpha = 1, \boldsymbol{\beta} = \mathbf{0}$ and $L([u_h]) = \eta_e r_e([u_h])$, it is the method of Brezzi et. al.;
5. if $\alpha = 0, \boldsymbol{\beta} = -\mathbf{n}_K$ and $L([u_h]) = \frac{\eta}{h_e}[u_h]$, it is the NIPG method;

Table Some DG schemes and their numerical fluxes.

Schemes	$\widehat{u_h}$	$\widehat{\mathbf{p}_h}$
1. IP	$\{u_h\}$	$\{\nabla_h u_h\} - \eta h^{-1}[u_h]$
2. NIPG	$\{u_h\} + \mathbf{n}_K \cdot [u_h]$	$\{\nabla_h u_h\} - \eta h^{-1}[u_h]$
3. LDG	$\{u_h\} - \boldsymbol{\beta} \cdot [u_h]$	$\{\mathbf{p}_h\} + \boldsymbol{\beta}[\mathbf{p}_h] - \eta h^{-1}[u_h]$
4. Brezzi et al.	$\{u_h\}$	$\{\mathbf{p}_h\} - \eta_e r_e([u_h])$
5. Bassi et al.	$\{u_h\}$	$\{\nabla_h u_h\} - \eta_e r_e([u_h])$

15.7.2 HDG formulation

Recall that

$$\begin{aligned}
&\int_K \mathbf{p}_h \cdot \mathbf{w}_h \, dx = -\int_K u_h \, \nabla_h \cdot \mathbf{w}_h \, dx + \int_{\partial K} \widehat{u_h} \, n_K \cdot \mathbf{w}_h \, ds = \int_K \nabla_w u_h \cdot \mathbf{w}_h \, dx \\
&\int_K \mathbf{p}_h \cdot \nabla_h \mathbf{v}_h \, dx = \int_K f \, v_h \, dx + \int_{\partial K} \widehat{\mathbf{p}_h} \cdot \, n_K \, v_h \, ds.
\end{aligned}$$

where we define

(15.120)
$$\widehat{\mathbf{p}}_h \cdot \mathbf{n} = \mathbf{p}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h) \quad \text{on } \partial K.$$

There are three key ideas for HDG.

1. Introduce appropriate approximation of u_h on ∂K, i.e., we have a new additional function $\widehat{u_h} \in M_h(\mathcal{E}_h)$.
2. Define $\widehat{\mathbf{p}}_h \cdot \mathbf{n} = \mathbf{p}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$ on ∂K.
3. If $\widehat{u_h}$ is known, then u_h and $\mathbf{p}_h$ can be solved locally, i.e., we can eliminate the local D.O.F of u_h and $\mathbf{p}_h$, and get a global system only for $\widehat{u_h}$. To do this, for each face $e \in \mathcal{E}_h^0$, we take $\widehat{u_h} \in M_h(e)$, and determine $\widehat{u_h}$ by requiring that,

$$\langle [\widehat{\mathbf{p}}_h], \widehat{v}_h \rangle_e = 0, \quad \forall \widehat{v}_h \in M_h(e) \quad \text{if } e \in \mathcal{E}_h^0 \tag{15.121}$$

$$\widehat{u} = 0 \quad \text{if } e \in \mathcal{E}_h^\partial, \tag{15.122}$$

To add the equations over all the elements, the system can be given by

$$\sum_{K\in\mathcal{T}_h} (\mathbf{p}_h, \mathbf{w}_h)_K + \sum_{K\in\mathcal{T}_h} (u_h, \nabla \cdot \mathbf{w}_h)_K - \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h, \mathbf{w}_h \cdot \mathbf{n} \rangle_{\partial K} = 0, \tag{15.123}$$

$$\sum_{K\in\mathcal{T}_h} (\mathbf{p}_h, \nabla v_h)_K - \sum_{K\in\mathcal{T}_h} \langle \widehat{\mathbf{p}}_h \cdot \mathbf{n}, v_h \rangle_{\partial K} = \sum_{K\in\mathcal{T}_h} (f, v_h)_K, \tag{15.124}$$

$$\sum_{K\in\mathcal{T}_h} \langle [\widehat{\mathbf{p}}_h], \widehat{v}_h \rangle_{\partial K} = 0 \tag{15.125}$$

Take $\widehat{\mathbf{p}}_h \cdot \mathbf{n} = \mathbf{p}_h \cdot \mathbf{n} + \tau(u_h - \widehat{u}_h)$ into it, and by some manipulation, we have

$$\sum_{K\in\mathcal{T}_h} (\mathbf{p}_h, \mathbf{w}_h)_K + \sum_{K\in\mathcal{T}_h} (u_h, \nabla \cdot \mathbf{w}_h)_K - \sum_{K\in\mathcal{T}_h} \langle \widehat{u}_h, \mathbf{w}_h \cdot \mathbf{n} \rangle_{\partial K} = 0, \tag{15.126}$$

$$-\sum_{K\in\mathcal{T}_h} (\nabla \cdot \mathbf{p}_h, v_h)_K - \sum_{K\in\mathcal{T}_h} \tau \langle u_h, v_h \rangle_{\partial K} + \sum_{K\in\mathcal{T}_h} \tau \langle \widehat{u}_h, v_h \rangle_{\partial K} = \sum_{K\in\mathcal{T}_h} (f, v_h)_K, \tag{15.127}$$

$$\sum_{K\in\mathcal{T}_h} \langle \mathbf{p}_h \cdot \mathbf{n}, \widehat{v}_h \rangle_{\partial K} + \sum_{K\in\mathcal{T}_h} \tau \langle u_h, \widehat{v}_h \rangle_{\partial K} - \sum_{K\in\mathcal{T}_h} \tau \langle \widehat{u}_h, \widehat{v}_h \rangle_{\partial K} = 0. \tag{15.128}$$

15.7.3 Consistency, boundedness and stability of DG methods

Lemma 113 (Consistency). *The DG scheme* (15.114) *with bilinear form* (15.116) *is consistent, that is*

$$B_h(u, v_h) = \int_\Omega f\, v_h\, dx \quad \forall\, v_h \in H^2(\mathcal{T}_h), \tag{15.129}$$

Proof. Note that $[u] = 0$ on $\mathcal{E}_h$, by (15.116), we have

$$\begin{aligned} B_h(u, v_h) :&= \int_\Omega \nabla u \cdot \nabla_h v_h\, dx - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [v_h]\, ds \\ &= \int_\Omega -\Delta u\, v_h\, dx + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \nabla u \cdot \mathbf{n}_K v_h\, ds - \int_{\mathcal{E}_h} \{\nabla_h u\} \cdot [v_h]\, ds \\ &= \int_\Omega f\, v_h\, dx, \end{aligned} \tag{15.130}$$

which completes the proof. □

With the consistency (16.4), and the DG scheme (15.114), it is easy to see that the Galerkin orthogonality holds

$$B_h(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h, \tag{15.131}$$

To consider the boundedness and stability of the bilinear form B_h, let $V(h) = V_h + H^2(\Omega) \cap H_0^1(\Omega) \subset H^2(\mathcal{T}_h)$, and define seminorms and norm for $v \in V(h)$ by the following relations:

$$|v|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} |v|_{1,K}^2, \quad |v|_{1,*}^2 = \sum_{e\in\mathcal{E}_h} h_e^{-1}\|[v]\|_{0,e}^2$$

$$\|v\|_*^2 = |v|_{1,h}^2 + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + |v|_{1,*}^2. \tag{15.132}$$

That (16.6) defines a norm can be seen from the next inequality (Poincare type inequality):

$$\|v\|_0 \le C(|v|_{1,h}^2 + |v|_{1,*}^2)^{1/2} \le C\,\|v\|_* \quad \forall\, v \in V(h). \tag{15.133}$$

Notice that the norm $\|v\|_*$ is the good to obtain boundedness of the bilinear form B_h, and the weaker norm $\|v\|_w = (|v|_{1,h}^2 + |v|_{1,*}^2)^{1/2}$ is the natural choice for analyzing the stability of DG methods.

Lemma 114. *Let $e \in \partial K$, then for any $v \in H^1(K)$, we have*

$$\|v\|_{0,e}^2 \le C(h_K^{-1}\|v\|_{0,K}^2 + h_K|v|_{1,K}^2). \tag{15.134}$$

Proof. Given K, let $\mathbf{n}$ be the unit outward normal vector on ∂K. Let $\boldsymbol{\rho}$ be vector field on $K \cup \partial K$ such that $\boldsymbol{\rho}\cdot\mathbf{n} \ge c_0$, where $c_0 > 0$. Then,

$$\begin{aligned} c_0 \int_{\partial K} v^2\, ds &\le \int_{\partial K} v^2 \boldsymbol{\rho}\cdot\mathbf{n}\, ds = -\int_K \nabla\cdot(v^2\boldsymbol{\rho})\, dx \\ &= -\int_K \nabla\cdot(v\nabla v)\cdot\boldsymbol{\rho}\, dx - \int_K v^2\nabla\cdot\boldsymbol{\rho}\, dx \\ &\le \|\boldsymbol{\rho}\|_{\infty,K}\int_K |v||\nabla v|\, dx + \|\nabla\cdot\boldsymbol{\rho}\|_{\infty,K}\int_K v^2\, dx \end{aligned}$$

Let (x_0, y_0) be the center of the inscribed circle of element K, then choose $\boldsymbol{\rho} = (x - x_0, y - y_0)$. It is easy to check that $\|\boldsymbol{\rho}\|_{\infty,D} \le h_K$, $\|\nabla\cdot\boldsymbol{\rho}\|_{\infty,D} \le 2$, and $c_0 \ge r$, where r is the radius of the inscribed circle of element K. Because the element K is shape regular, i.e., h_K can be bounded by r, then by Schwarz inequality, we finish the proof. □

With this trace inequality, we have the following results.

Lemma 115 (Boundedness). *For $1 \le j \le 9$, $B_h = B_h^{(j)}$ satisfies*

$$B_h(u, v) \le C_b\|u\|_*\,\|v\|_* \quad \forall\, u, v \in V(h), \tag{15.135}$$

where C_b is a positive constant depending on the angle condition, the polynomial degree, an upper bound on the edge-dependent penalty parameter η for the methods that contain the penalty term α^j or α^r and, in the case of the LDG method, an upper bound for the coefficient β.

Proof. Let us take IP method as an example to show the proof.

It is ease to see that

$$\int_\Omega \nabla_h w\cdot\nabla_h v\, dx \le |w|_{1,h}|v|_{1,h}$$

and

$$\int_{\mathcal{E}_h} \frac{\eta}{h_e}[w]\cdot[v_h]\, ds \le C|w|_{1,*}|v|_{1,*}$$

Let us consider the term $\int_{\mathcal{E}_h}\{\nabla_h w\}\cdot[v]\,ds$ and another term $\int_{\mathcal{E}_h}[w]\cdot\{\nabla_h v_h\}\,ds$ can be analyzed in the same way.

$$\begin{aligned}\int_{\mathcal{E}_h}\{\nabla_h w\}\cdot[v]\,ds &= \sum_{\mathcal{E}_h}\int_e (h_e^{1/2}\{\nabla_h w\})\cdot(h_e^{-1/2}[v])\,ds\\ &\le C\left[\sum_{K\in\mathcal{T}_h}(|w|_{1,k}^2 + h_K^2|u|_{2,k}^2)\right]^{1/2}\left[\sum_{e\in\mathcal{E}_h} h_e^{-1}\int_e |[v]|^2 ds\right]^{1/2}\\ &\le C\|w\|_*|v|_*\end{aligned}$$

In the above inequalities, we apply the trace inequality (15.134). So we prove

$$B_{1,h}^{(1)}(w,v)\le C_b\|w\|_*\|v\|_*$$

□

For the stability, we have the following results.

Lemma 116 (Stability). *For* $1\le j\le 7$, $B_h = B_h^{(j)}$ *satisfies*

$$B_h(v,v)\ge C_s\|v\|_*^2 \quad \forall\, v\in V_h, \tag{15.136}$$

if $\eta_0 = \inf_e \eta_e > 0$ *for the methods with* $j = 2,3,4,6,7$, $\eta_0 > 3$ *for the method with* $j = 5$, η_0 *is large enough for IP method* $(j = 1)$, *where* C_s *is a positive constant depending on the angle condition, the polynomial degree, a bound on the edge-dependent penalty parameter* η *and, in the case of the LDG method, a bound for the coefficient* β.

Proof. Again, let us consider IP method as an example to prove this lemma.

From the proof of the boundedness, we know that

$$\int_{\mathcal{E}_h}\{\nabla_h v\}\cdot[v]\,ds\le C\|v\|_*|v|_{1,*}$$

Then

$$\begin{aligned}B_{1,h}^{(1)}(v,v) &\ge |v|_{1,h}^2 + \eta_0|v|_{1,*}^2 - C\|v\|_*|v|_{1,*}\ge |v|_{1,h}^2 + \eta_0|v|_{1,*}^2 - C(\epsilon\|v\|_*^2 + \frac{|v|_{1,*}^2}{4\epsilon})\\ &= |v|_{1,h}^2 + (\eta_0 - \frac{C}{4\epsilon})|v|_{1,*}^2 - C\epsilon\|v\|_*^2,\end{aligned}$$

where the $0<\epsilon<1$ is an arbitrary constant number. If we let ϵ is very very small, then the last term can be ignored. Hence,

$$B_{1,h}^{(1)}(v,v)\ge C_s(|v|_{1,h}^2 + |v|_{1,*}^2) = C_s\|v\|_w^2, \tag{15.137}$$

If we choose η_0 be large enough. □

Notice that (16.11) only claims the coercivity of the bilinear form B_h on V_h. Lack of coercivity of B_h on V is one source of difficulty in studying the DG methods.

15.7.4 Approximation and error estimates

We now turn to error estimations for the DG methods. Write the error as

$$e = u - u_h = (u - u_I) + (u_I - u_h),$$

where $u_I \in V_h$ is a suitable interpolant of the exact solution. If u_I is chosen to be the usual continuous piecewise polynomial interpolant, then the jumps of $(u - u_I)$ will be zero at the interelement boundaries. Then

$$\|u - u_I\|_*^2 = |u - u_I|_{1,h}^2 + \sum_{K \in \mathcal{T}_h} h_K^2 |u - u_I|_{2,K}^2 + \sum_{e \in \mathcal{E}_h^\partial} h_e^{-1} \|[u - u_I]\|_{0,e}^2 \le C_a^2 h^{2p} |u|_{p+1,\Omega}^2. \tag{15.138}$$

To analyze the method of Baumann-Oden ($j = 8$) and extend the analysis to nonconforming meshes, it is convenient to take an interpolant u_I which is discontinuous across the interelement boundaries. We just require the local approximation property

$$|u - u_I|_{s,K} \le C h_K^{p+1-s} |u|_{p+1,K};$$

then for the global approximation error, we have

$$\|u - u_I\|_* \le C_a h^p |u|_{p+1,\Omega}. \tag{15.139}$$

Theorem 117 (Strang's Lemma). *Let u and u_h be the solutions of* (15.1) *and* (15.114) *respectively. Assume $u \in H^{p+1}(\Omega)$. Then for all the DG methods, we have*

$$\|u - u_h\|_* \le (1 + \frac{C_b}{C_s}) \inf_{v_h \in V_h} \|u - v_h\|_* + \frac{1}{C_s} \sup_{w_h \in V_h} \frac{|B_h(u, w_h) - (f, w_h)|}{\|w_h\|_*}. \tag{15.140}$$

Proof. Let v_h be the usual continuous piecewise linear interpolant of u. Recall the boundedness and stability of the bilinear form B_h. We have

$$\begin{aligned} C_s \|v_h - u_h\|_*^2 &\le B_h(v_h - u_h, v_h - u_h) \\ &= B_h(v_h - u, v_h - u_h) + B_h(u - u_h, v_h - u_h), \\ &\le C_b \|v_h - u\|_* \|v_h - u_h\|_* + B_h(u - u_h, v_h - u_h) \end{aligned}$$

So

$$\|v_h - u_h\|_* \le \frac{C_b}{C_s} \|v_h - u\|_* + \frac{1}{C_s} \frac{B_h(u - u_h, v_h - u_h)}{\|v_h - u_h\|_*} \tag{15.141}$$

Furthermore,

$$\begin{aligned} \|u - u_h\|_* &\le \|u - v_h\|_* + \|v_h - u_h\|_* \\ &\le (1 + \frac{C_b}{C_s}) \|u - v_h\|_* + \frac{1}{C_s} \frac{B_h(u - u_h, v_h - u_h)}{\|v_h - u_h\|_*} \\ &\le (1 + \frac{C_b}{C_s}) \|u - v_h\|_* + \frac{1}{C_s} \sup_{w_h \in V_h} \frac{|B_h(u - u_h, w_h)|}{\|w_h\|_*} \end{aligned}$$

which finishes the proof. □

Note that for the consistent DG methods, $B_h(u, v_h) = (f, v_h)$, then

$$\|u - u_h\|_* \le (1 + \frac{C_b}{C_s}) \inf_{v_h \in V_h} \|u - v_h\|_* \le (1 + \frac{C_b}{C_s}) \|u - u_I\|_* \lesssim h^p$$