11

Non-Conforming Finite Element Methods

11.1 Drawbacks of conforming finite element methods

It is evident that the conforming finite element is a natural way to construct an approximation space for a given Sobolev space. However, because of some essential character of the conforming element, it is far from perfect in practical use. For example, we take the two aspects below.

11.1.1 Difficulty in construction

In general, it is not straightforward to construct a subspace that consists of piecewise polynomials for a given Sobolev space. Difficulties can come from different sources, we only take two examples.

1. When we consider to discretize the second order problem on triangular/rectangular grids, the linear/-bilinear element are well-known and user-friendly conforming finite elements. However, it is proved to be impossible to construct a nontrivial continuous piecewise linear polynomial space or a 4-nodal-parameter element space that contains linear polynomials on a general quadrilateral grid.
2. When we consider the discretization of the fourth order problem, a conforming finite element space consists of piecewise polynomial functions that are globally C^1 continuous. For a piecewise P_k function w_h, in order that w_h is continuous and C^1 continuous across the internal edges, $k+1$ restriction about its evaluation and k restrictions about its normal derivative have to be equipped on w_h at every edge. This set of restrictions are generally overdetermined for a k-th degree polynomial on a triangle or a quadrilateral if k is not sufficiently big.

   Indeed, in order to construct a globally C^1 piecewise polynomial space on a simplex grid, polynomials of degree at least 5 have to involved in 2D [**?**, **?**] and at least 9 in 3D [**?**].

11.1.2 Restrictions in applications

If we focus ourselves in conforming elements, extra inflexibility can be met in practical applications. The $(P_k)^2 - P_{k-1}$ element for Stokes problem on triangular grids is a typical example. Here the continuous piecewise P_k element space is the conforming approximation of $H_0^1(\Omega)$ and piecewise P_{k-1} element space is the conforming approximation of $L_0^2(\Omega)$. The finite element pair is proved to be stable on general grids without singular vertex when $k \geqslant 4$ [**?**, **?**], but if k is smaller than 4, namely $k = 2, 3$, the finite element pair is stable only on some quite special grids.

11.2 Nonconforming finite element methods

A methodology to overcome some difficulties related to the conforming element is the nonconforming finite element. The nonconforming finite element is indeed a methodology to construct piecewise polynomial functions which are of looser restriction on continuity than conforming element functions, and to admit external approximation for a certain Sobolev space. We take the nonconforming finite element method for second and fourth order elliptic problems to illustrate the methodology.

Through this section, let $\Omega \subset \mathbb{R}^d$ be a bounded domain.

11.2.1 Nonconforming finite element method for second order problem

The variational problem to be discretized is: find $u \in H_0^1(\Omega)$, such that

$$a(u, v) := \int_\Omega \nabla u \nabla v = \int_\Omega f v := (f, v), \ \forall\, v \in H_0^1(\Omega).$$

This is the variational form of the Dirichlet boundary value problem of Poisson equation.

Let $\mathcal{T}_h$ be a triangulation of Ω. Define

$$a_h(v_h, w_h) := \sum_{T\in\mathcal{T}} \int_T \nabla u_h \nabla v_h, \quad |w_h|_h := a_h(w_h, w_h)^{1/2},$$

where v_h and w_h are piecewise polynomials on $\mathcal{T}_h$.

Let V_h be a space of piecewise polynomials, which is not necessarily a subspace of H^1, such that $|\cdot|$ is a norm on V_h. The finite element problem is to find $u_h \in V_h$, such that

$$a_h(u_h, v_h) = (f, v_h), \ \forall\, v_h \in V_h. \tag{11.1}$$

Note that when $V_h \subset H_0^1(\Omega)$, this is exactly the conforming finite element problem.

Finite element spaces

To formulate a finite element discretization, the main task is to construct a finite element space V_h, or equivalently a finite element. There have been many successful finite elements for the model problem. We introduce three here.

1. The nonconforming P_1 element on triangles, which is also known as the Crouzeix-Raviart element [18]. The finite element is defined by (T, P_T, D_T) with
   - T is a triangle;
   - $P_T = P_1(T)$;
   - $D_T = \{d_i\}_{i=1}^3$ with for any $v \in H^1(Q)$, $d_i(v) = \int_{e_i} v$, where e_i is the edge of T, $i = 1 : 3$.

   Associated with this finite element, a finite element space is defined on $\mathcal{T}_h$ as

   $$W_h := \{w_h \in L^2(\Omega) : w_h|_T \in P_1(T), \ \forall\, T \in \mathcal{T}_h, \ \fint_e w_h \text{ is continuous across the interior edge } e\}.$$

   In regard to the boundary condition of the boundary value problem, the well-posedness of the finite element problem and the convergence issue, define the finite element space V_h by

   $$V_h := \{v_h \in W_h : \fint_e v_h = 0, \text{ at boundary edge } e\}.$$

2. The rotated Q_1 element on rectangle. The finite element is defined by (Q, P_Q, D_Q) with

- Q is a rectangle;
- $P_Q = P_1(Q) + \mathrm{span}\{x^2 - y^2\}$, x, y being the local coordinates of Q;
- $D_Q = \{d_i\}_{i=1}^4$: for $v \in H^1(Q)$, $d_i(v) = \fint_{e_i} v$, where e_i is the edge of Q, $i = 1:4$.

Associated with this finite element, a finite element space can be defined on $\mathcal{T}_h$ as

$$W_h := \{w_h \in L^2(\Omega) : w_h|_T \in P_Q(Q),\ \forall\, Q \in \mathcal{T}_h,\ \fint_e w_h \text{ is continuous across the interior edge } e\}.$$

Again, in regard to the boundary condition of the boundary value problem, the well-posedness of the finite element problem and the convergence issue, we define the finite element space V_h by

$$V_h := \{v_h \in W_h : \fint_e v_h = 0, \text{ at boundary edge } e\}.$$

3. Nonconforming P_1 element on quadrilaterals [**?**]. Similar to the previous two cases but differently, we just define a finite element space on a quadrilateral grid by

$$W_h := \{w_h \in L^2(\Omega) : w_h|_Q \in P_1(Q),\ \forall\, Q \in V_h,\ \fint_e w_h \text{ is continuous across the interior edge } e\}.$$

While to be in accordance with the boundary conditions, define

$$V_h := \{v_h \in W_h : \fint_e v_h = 0, \text{ at boundary edge } e\}.$$

In order to practical application, the finite element space has to be described clearly in its dimension and local basis functions.

Given any vertex v of the grid, denote by $E(v)$ the set of all edges e such that one of the endpoints is v. We can choose $\varphi_v \in W_h$ be such that

$$\fint_e \varphi_v = \begin{cases} 1, & e \in E_v \\ 0, & \text{else.} \end{cases} \tag{11.2}$$

See Figure 11.1 for an illustration of such a function. Then the finite element space can be spanned by these function associated with the vertices. Indeed, we can prove the lemma below. Denote by $\mathcal{N}_h$ the set of vertices, and by $\mathcal{N}_h^i$ the set of interior vertices.

The basic observation for this finite element is that: for a general quadrilateral Q, as shown in Figure 11.2, if we connect the midpoints of the four edges sequentially, then a parallelogram will be formed. Thus for p defined on Q,

$$p \text{ is a linear polynomial } \Leftrightarrow\ p(m_1) + p(m_3) = p(m_2) + p(m_4).$$

Lemma 82. *For any $v \in \mathcal{N}_h$, denote by $\varphi_v \in W_h$ the function defined by* (11.2).

a) $\dim(W_h) = \#\mathcal{N}_h - 1$. *Moreover, for any vertex $v_0 \in \mathcal{N}_h$, $\{\varphi_v\}_{v\in\mathcal{N}_h, v\neq v_0}$ forms a basis of W_h.*

b) $\dim(V_h) = \#\mathcal{N}_h^i$. *Moreover, $\{\varphi_v\}_{v\in\mathcal{N}_h^i}$ forms a basis for V_h.*

Lemma 82 gives the degrees of freedom in practical programing and computation.

Remark 18. On general convex quadrilateral grids, we can still define a finite element with the same nodal parameters as that of the rotated Q^1 element, and a set of shape functions which are just that of rotated Q^1 element on rectangles as special convex quadrilaterals. Namely, a rotated Q^1-type element can be defined on convex quadrilaterals. Note that there is no finite element alike to the bilinear element defined on quadrilateral grid.

Remark 19. On quadrilateral grid, the conforming piecewise linear polynomial space is a trivial subspace of $H^1(\Omega)$. Indeed, it does not provide degree of freedoms associated with interior vertices. Particularly, the continuous piecewise linear polynomial subspace of $H_0^1(\Omega)$ is exactly $\{0\}$. By the nonconforming way, a nontrivial consistent approximation of $H^1(\Omega)$ with piecewise linear polynomial is constructed.

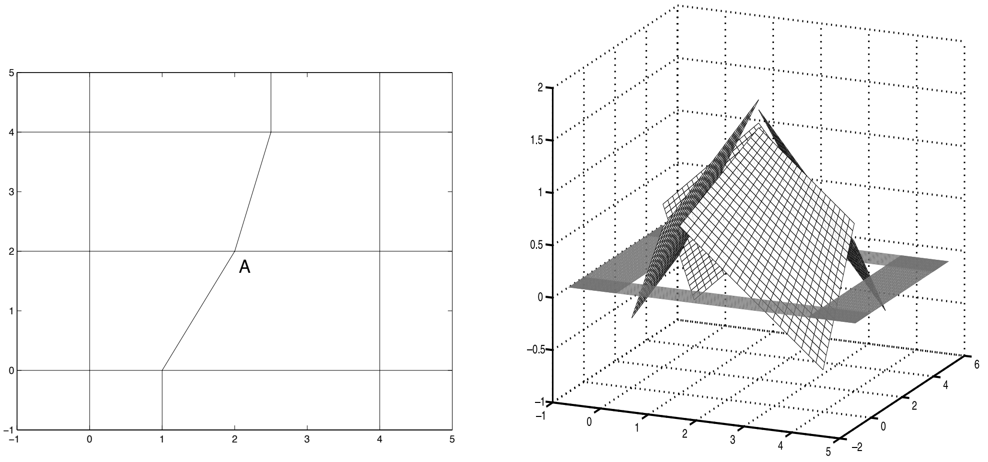


Fig. 11.1. Illustration of the nodal basis function. Associated with the vertex A shown in the left figure, the basis function φ_A has the profile as shown in the right figure.

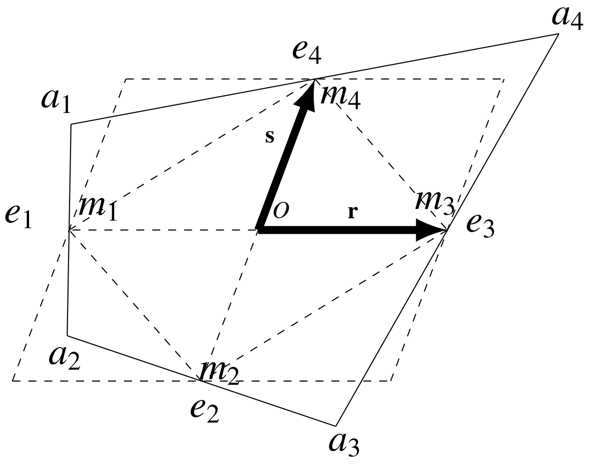
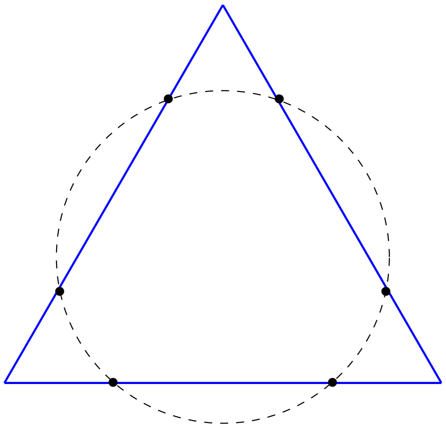


Fig. 11.2. Illustration of a convex quadrilateral.

11.2.2 Lack of high order nonconforming elements

It is difficult to construct 2nd order nonconforming finite element spaces for 2nd order elliptic problem. Here is a counter example.

1. the Gauss points are located **symmetrically** on the boundary.
2. $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$ is **symmetric** on the triangle
3. Gauss points lie on the contours of $\lambda_1^2 + \lambda_2^2 + \lambda_3^2$
4. $\phi := 2 - 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$ vanish on the Gauss points.



11.2.3 Nonconforming finite element method for fourth order problem

In contrast to for second order problems, the nonconforming finite element methods play more important roles for problems of higher order. We take the fourth order problem for an illustration.

We consider the boundary value problem for biharmonic equation:

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega \\ u = 0, \ \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases} \tag{11.3}$$

This model problem, which is used for modeling in elastic mechanics and fluid mechanics, is frequently encountered in applied sciences.

The variational problem is to find $u \in H_0^2(\Omega)$, such that

$$a(u,v) := \int_\Omega \nabla^2 u : \nabla^2 v = (f,v) \ \ \forall\, v \in H_0^2(\Omega).$$

Here, the symbol ":" denotes the contraction of two tensors. For simplicity, let $\alpha,\beta \in \mathbb{R}^{m\times n}$, then $\alpha : \beta = \sum_{1\leqslant m, 1\leqslant j\leqslant n} \alpha_{ij}\beta_{ij}$.

Let $\mathcal{T}_h$ be a triangulation of Ω. Similar to the second order problem, we define

$$a_h(v_h, w_h) := \sum_{T\in\mathcal{T}} \int_T \nabla^2 v_h \nabla^2 w_h, \ \ |w_h|_h := a_h(w_h,w_h)^{1/2},$$

where v_h and w_h are piecewise polynomials on $\mathcal{T}_h$.

Let V_h be a space of piecewise polynomials, which is not necessarily a subspace of H^2, such that $|\cdot|_h$ is a norm on V_h. The finite element problem is to find $u_h \in V_h$, such that

$$a_h(u_h, v_h) = (f, v_h), \ \ \forall\, v_h \in V_h. \tag{11.4}$$

Finite element spaces

Again, we present two examples of the many successful nonconforming finite elements for fourth order problem.

1. The nonconforming P_2 element, which is known as the Morley element. This is the most famous nonconforming finite element for fourth order problem. The finite element is defined by (T, P_T, D_T) with

- T is a triangle;
- $P_T = P_2(T)$;
- $D_T = \{d_i, d_{3+i}\}_{i=1}^3$: for $v \in H^2(Q)$, $d_i(v) = v(a_i)$ and $d_{3+i}(v) = \fint_{e_i} \frac{\partial v}{\partial \mathbf{n}}$, where a_i is the vertex of T and e_i is the edge of T, $i = 1 : 3$.

Associated with this finite element, a finite element space can be defined on $\mathcal{T}_h$ as

$$W_h := \{w_h \in L^2(\Omega) : w_h|_T \in P_2(T),\ \forall\, T \in \mathcal{T}_h,\ w_h \text{ is continuous at vertices,} \quad \text{and } \fint_e \frac{\partial v}{\partial \mathbf{n}} \text{ is continuous across the interior edge } e\}.$$

In regard to the boundary condition of the boundary value problem, the well-posedness of the finite element problem and the convergence, we define the finite element space V_h by

$$V_h := \{w_h \in V_h : w_h \text{ vanishes at boundary vertices, and } \fint_e \frac{\partial v}{\partial \mathbf{n}} \text{ vanishes at boundary edge } e\}.$$

2. The rectangle Morley element. The finite element is defined by (Q, P_Q, D_Q) with
   - Q is a rectangle;
   - $P_Q = P_2(Q) + \text{span}\{x^3, y^3\}$, x, y being the local coordinates of Q;
   - $D_Q = \{d_i, d_{4+i}\}_{i=1}^4$: for $v \in H^2(Q)$, $d_i(v) = v(a_i)$ and $d_{4+i}(v) = \fint_{e_i} \frac{\partial v}{\partial \mathbf{n}}$, where a_i is the vertex of T and e_i is the edge of T, $i = 1 : 4$.

   Associated with this finite element, a finite element space can be defined on $\mathcal{T}_h$ as

$$W_h := \{w_h \in L^2(\Omega) : w_h|_T \in P_Q(T),\ \forall\, Q \in \mathcal{T}_h,\ w_h \text{ is continuous at vertices,} \quad \text{and } \fint_e \frac{\partial v}{\partial \mathbf{n}} \text{ is continuous across the interior edge } e\}.$$

In regard to the boundary condition of the boundary value problem, the well-posedness of the finite element problem and the convergence, we define the finite element space V_h by

$$V_h := \{w_h \in V_h : w_h \text{ vanishes at boundary vertices, and } \fint_e \frac{\partial v}{\partial \mathbf{n}} \text{ vanishes at boundary edge } e\}.$$

Remark 20. On general convex quadrilateral grids, we can still define a finite element with the same nodal parameters as that of the rectangle Morley element, and a set of shape functions which are just that of rectangle Morley element on rectangles — special convex quadrilaterals.

Remark 21. In contrast to the construction of conforming finite element spaces where at least fifth degree polynomials have to be involved, the nonconforming Morley element is constructed with quadratic polynomials involved only and much fewer nodal parameters. Indeed, the Morley element space provides the minimal order consistent approximations to $H^2(\Omega)$.

11.3 MWX element and generalization

In this section, we will consider the nonconforming finite element methods for $2m$-th order elliptic equation.

We first introduce some basic notation. Given a nonnegative integer k and a bounded domain $G \subset R^d$ with boundary ∂G, let $H^k(G)$, $H_0^k(G)$, $(\cdot,\cdot)_{k,G}$, $\|\cdot\|_{k,G}$ and $|\cdot|_{k,G}$ denote the usual Sobolev spaces, inner product, norm and semi-norm respectively.

For an d dimensional multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$, define

$$|\alpha| = \sum_{i=1}^{d} \alpha_i, \quad \partial^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

We will use α,β,γ to denote the multi-indexes. Let e_i denote the corresponding dimensional multi-indexes with the i-th component 1 and the others 0. For $k \geqslant 1$, let A_k be the set consisting of all multi-index α with $\sum_{i=k+1}^{d} \alpha_i = 0$.

Let b_0 be nonnegative constant and b_α be positive constants,$|\alpha| = m$. Define

$$a(v, w) = \int_\Omega \Big(\sum_{|\alpha|=m} b_\alpha \partial^\alpha v\, \partial^\alpha w + b_0 v w \Big), \quad \forall v, w \in H^m(\Omega). \tag{11.5}$$

Let W be $H_0^m(\Omega)$ or $H^m(\Omega)$, and let $f \in L^2(\Omega)$. We consider the following variational problem: find $u \in W$ such that

$$a(u, v) = (f, v), \quad \forall v \in W. \tag{11.6}$$

We assume that problem (11.6) has unique solution for any $f \in L^2(\Omega)$.

The above variational problem corresponds to the following $2m$-th order partial differential equation:

$$\sum_{|\alpha|=m} (-1)^{|\alpha|} \partial^\alpha (b_\alpha \partial^\alpha u) + b_0 u = f, \quad \text{in } \Omega. \tag{11.7}$$

When $W = H_0^m(\Omega)$, the variational problem (11.6) corresponds to the homogeneous Dirichlet boundary problem of partial equation (11.7) with boundary conditions:

$$\frac{\partial^k u}{\partial \nu^k}\Big|_{\partial\Omega} = 0, \quad 0 \le k \le m-1, \tag{11.8}$$

where $\nu = (\nu_1, \nu_2, \cdots, \nu_d)^\top$ is the unit outer normal to $\partial\Omega$.

When $W = H^m(\Omega)$, problem (11.6) corresponds to the boundary problem of (11.7) with some natural boundary conditions.

11.3.1 CR element on *d*-D

The Crouzeix-Raviart element in 2-D can be generated to arbitrary dimension. This is a unified family of finite elements which work for second order problems.

For a d-dimensional problem, the finite element is defined by (T, P_T, D_T) with

- T is a d-simplex;
- $P_T = P_1(T)$;
- $D_T = \{d_i\}_{i=1}^{d+1}$: for $v \in H^1(Q)$, $d_i(v) = \fint_{f_i} v$, where f_i is an d-subsimplex of T, $i = 1 : d+1$.

Note that $\dim(D_T) = \dim(P_T) = d + 1$, and the finite element is P_T-unisolvent.

Based on this finite element, a finite element space can be defined on $\mathcal{T}_h$ as

$$W_h := \{w_h \in L^2(\Omega) : w_h|_T \in P_1(T),\ \forall\, T \in \mathcal{T}_h,\ \fint_f w_h \text{ is continuous across any interior (d-1)-subsimplex } f\}.$$

In regard to the boundary condition of the boundary value problem, the well-posedness of the finite element problem and the convergence, we define the finite element space V_h by

$$V_h := \{v_h \in W_h : \fint_f v_h = 0, \text{ at any boundary (d-1)-subsimplex } f\}.$$

This generalization is quite natural and straightforward.

11.3.2 Morley element on d-D

The Morley element can be extended to arbitrary dimension [**?**]. This is a unified family of nonconforming P_2 element for fourth order problems.

The finite element is defined by (T, P_T, D_T) with

- T is a simplex;
- $P_T = P_2(T)$;
- $D_T = \{\varphi_i, \psi_j\}_{i=1:d(d+1)/2}^{j=1:d+1}$: for any $v \in H^2(Q)$, $\varphi_i(v) = \fint_{f_i} \frac{\partial v}{\partial \mathbf{n}}$, where f_i is a $(d-1)$-subsimplex of T for $i = 1 : (d+1)$, and $\psi_j(v) = \fint_{e_j} v$, where e_j is a $(d-2)$-subsimplex of T for $j = 1 : d(d+1)/2$.

Again, $\dim(D_T) = \dim(P_T) = (d+1)(d+2)/2$, and the finite element is P_T-unisolvent.

Based on this finite element, a finite element space can be defined on $\mathcal{T}_h$ as

$$W_h := \{w_h \in L^2(\Omega) : w_h|_T \in P_2(T),\ \forall\, T \in \mathcal{T}_h,\ \fint_e w_h \text{ is continuous across any (d-2)-subsimplex } e,$$
$$\text{and } \fint_f \frac{\partial v}{\partial \mathbf{n}} \text{ is continuous across any interior (d-1)-subsimplex } f\}.$$

In regard to the boundary condition of the boundary value problem, the well-posedness of the finite element problem and the convergence, we define the finite element space V_h by

$$V_h := \{w_h \in V_h : \fint_e w_h \text{ vanishes at any boundary (d-2)-subsimplex } e,$$
$$\text{and } \fint_f \frac{\partial v}{\partial \mathbf{n}} \text{ vanishes at any boundary (d-1)-subsimplex } f\}.$$

Here we prove the unisolvence of Morley element for $d = 3$.

Lemma 83. *Let $d = 3$, T be a tetrahedron, and F_i, $i = 1 : 4$ be the faces of T. Given $v \in C^1(T)$, the degrees of freedom given uniquely determine the integrals of all first order derivatives $\int_{F_i} \nabla v$.*

Proof. Given $1 \le j \le 4$, denote the unit normal of F_j by ν, the edges of F_j by $S_1, \ldots, S_3$, and the unit out normal of S_i by $\mathbf{n}^{(i)}$, viewed as the boundary of a triangle in $\mathbb{R}^2$. Given any constant vector $\alpha \in \mathbb{R}^3$, let $\tau = \alpha - (\alpha \cdot \nu)\mu$, then $\tau \cdot \nu = 0$, namely τ is tangent to F. It follows that

$$\int_{F_j} \nabla v \cdot \alpha = (\alpha \cdot \nu) \int_{F_j} \frac{\partial v}{\partial \nu} + \sum_{i=1}^{3} \tau \cdot \mathbf{n}^{(i)} \int_{S_i} v. \tag{11.9}$$

This gives an explicit expression of $\int_{F_j} \nabla v \cdot \alpha$ in terms of the degrees of freedom for any $\alpha \in \mathbb{R}^3$. The desired result then follows. □

It follows from Lemma 83 immediately that $\int_F \nabla v_h$ is continuous across any internal face F, and vanishes on any boundary face F.

Lemma 84. *The Morley element is unisolvent.*

Proof. Because the dimension of $P_2(T)$ and the number of degrees of freedom are both 10, it is enough to show that if $p \in P_2(T)$, and

$$\int_{e_j} p = 0,\ 1 \le j \le 6,\ \int_{f_j} \frac{\partial p}{\partial \nu} = 0,\ 1 \le j \le 4,$$

then $p \equiv 0$. By Lemma 83 and its proof, we deduce that

$$\int_{F_j} \nabla p = 0, \ 1 \le j \le 4. \tag{11.10}$$

Now let $1 \le k, j \le 3$. By Green's formula and (11.10) we have

$$\frac{\partial^2 p}{\partial x_k \partial x_l} = \frac{1}{|T|}\int_T \frac{\partial^2 p}{\partial x_k \partial x_l} = \frac{1}{|T|}\sum_{j=1}^{4}\int_{F_j} \frac{\partial p}{\partial x_k}\nu_l = 0.$$

That is, $p \in P_1(T)$. From (11.10), $\nabla p = 0$ and p is a constant on T. Hence $p \equiv 0$. □

11.3.3 MWX element for 2*m*-th order problems in $\mathbb{R}^d$ when $d \ge m \ge 1$

In this section, we construct a minimal piecewise polynomial approximations to $H^m(\Omega)$ for $\Omega \subset R^d$ with $d \geqslant m \geqslant 1$, named Morley-Wang-Xu element [**?**]. We can see that both the Crouzeix-Raviart element and the Morley element are special cases of this unified family.

Given a d-simplex T with vertices a_i, $1 \leqslant i \leqslant d+1$, let $\lambda_1, \lambda_2, \cdots, \lambda_{d+1}$ be the barycentric coordinates of T. For $1 \leqslant k \leqslant d$, let $\mathcal{F}_{T,k}$ be the set consisting of all $(d-k)$-dimensional subsimplexes of T. For any F in $\mathcal{F}_{T,k}$, let $|F|$ denote its measure, and let $\nu_{F,1}, \cdots, \nu_{F,k}$ be its unit outer normals which are linearly independent.

Again, we give the description of (T, P_T, D_T) for the unified family of finite elements. For the m-th order problem in d-D, T is a d-simplex and $P_T = P_m(T)$. The set of degrees of freedom, denoted by D_T^m, will be given in the following.

For $1 \le k \le m$, any $(d-k)$-dimensional subsimplex $F \in \mathcal{F}_{T,k}$ and $\beta \in A_k$ with $|\beta| = m-k$, define

$$d_{T,F,\beta}(v) = \frac{1}{|F|}\int_F \frac{\partial^{|\beta|} v}{\partial \nu_{F,1}^{\beta_1} \cdots \nu_{F,k}^{\beta_k}}, \quad \forall v \in H^m(T). \tag{11.11}$$

By the Sobolev embedding theorems [**?**], $d_{T,F,\beta}$ is a continuous linear functional on $H^m(T)$. Then the set of the degrees of freedom is given by

$$D_T^m = \{\, d_{T,F,\beta} \ : \ \beta \in A_k \text{ with } |\beta| = m-k, \ F \in \mathcal{F}_{T,k}, \ 1 \le k \le m\}. \tag{11.12}$$

That is, the degrees of freedom are the integral averages of normal derivatives of order $m-k$ on all subsimplexes of dimension $n-k$ for $1 \le k \le m$.

For nonnegative integers i, j with $i \le j$, set the combinational number $C_j^i = \dfrac{j!}{i!(j-i)!}$. Then for each $1 \le k \le m$, T has C_{n+1}^{n-k+1} subsimplexes of $(n-k)$-dimension. For each $(n-k)$-dimensional subsomplex F, the number of all $(m-k)$-th order direction derivatives, with respect to $\nu_{F,1}, \cdots, \nu_{F,k}$, is C_{m-1}^{m-k}. Therefore, by the well-known Vandermonde combinatorial identity, the number of the total degrees of freedom is given by $\sum_{k=1}^{m} C_{n+1}^{n-k+1} C_{m-1}^{m-k} = C_{n+m}^m$ which is precisely the dimension of $P_m(T)$. The finite element is P_T-unisolvent.

The degrees of freedom in these cases are depicted in Table 11.1 for $m \leqslant d \leqslant 3$. For $m = 1$ and $d = 1$, we obtain the well-known conforming linear elements. This is the only conforming element in this family of elements. For $m = 1$ and $d \geqslant 2$, we obtain the well-known nonconforming linear elements. For $m = 2$, we recover the well-known Morley element for $d = 2$ and its generalization to $d \geqslant 2$. For $m = 3$ and $d = 3$, we obtain a new cubic element on a 3-simplex that has 20 degrees of freedom.

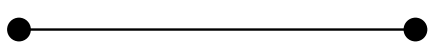
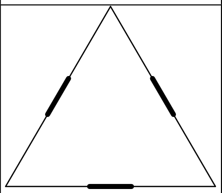
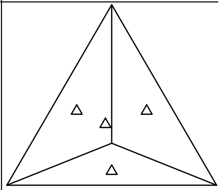
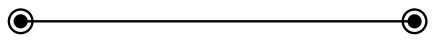
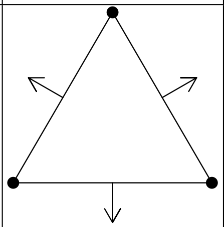
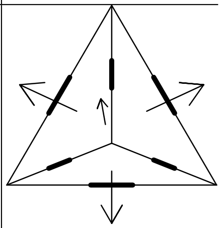
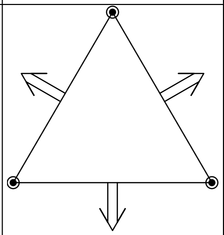
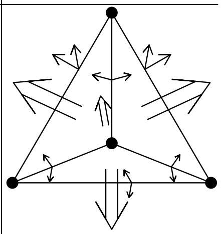
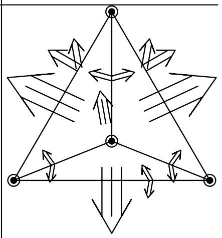
$m\backslash d$	1	2	3
1			
2			
3			
4			

Table 11.1. Degrees of freedom for the construction: $m \leq d + 1$

11.3.4 Nonconforming element when $m = d + 1$

It is extremely difficult to construct the convergent nonconforming element for the case in which $m \geq d$. Recently, Wu and Xu give the construction when $m = d + 1$, see [**?**].

For any simplex T, let q_T be the volume bubble function of the simplex K. Specifically, we have

$$q_T = \lambda_1 \lambda_2 \cdots \lambda_{n+1}.$$

The shape function space $P_T = P_T^{(m,n)}$ when $m = n + 1$ is defined as

$$P_T^{(n+1,n)} := \mathcal{P}_{n+1}(T) + q_T \mathcal{P}_1(T), \tag{11.13}$$

where $\mathcal{P}_k(T)$ denotes the space of all polynomials defined on T with a degree not greater than k, for any integer $k \geq 0$.

For $k \geq 1$, let A_k be the set consisting of all multi-indexes α with $\sum_{i=k+1}^{d} \alpha_i = 0$. For $1 \leq k \leq d$, any $(n-k)$-dimensional sub-simplex $F \in \mathcal{F}_{T,k}$ and $\alpha \in A_k$ with $|\alpha| = d + 1 - k$, we define

$$d_{T,F,\alpha}(v) := \frac{1}{|F|}\int_F \frac{\partial^{n+1-k} v}{\partial \nu_{F,1}^{\alpha_1}\cdots\partial \nu_{F,k}^{\alpha_k}} \qquad \forall v \in H^{d+1}(\Omega). \tag{11.14}$$

When $|\alpha| = 0$, we define

$$d_{T,a_i,0}(v) := v(a_i) \qquad \forall v \in H^{d+1}(\Omega). \tag{11.15}$$

By the Sobolev embedding theorem, $d_{T,F,\alpha}$ and $d_{T,a_i,0}$ are continuous linear functionals on $H^{d+1}(T)$. Then, the set of the degrees of freedom is

$$\begin{aligned} D_T^{(d+1,d)} :=\ &\{d_{T,F,\alpha}\ :\ \alpha \in A_k \text{ with } |\alpha| = d+1-k, F \in \mathcal{F}_{T,k}, 1 \le k \le d\} \\ &\cup \{d_{T,a_i,0}\ :\ 1 \le i \le d+1\}. \end{aligned} \tag{11.16}$$

We also number the local degrees of freedom by

$$d_{T,1}, d_{T,2}, \cdots, d_{T,J},$$

where J is the number of local degrees of freedom.

As a natural extension of MWX elements proposed in [**?**], the diagrams of the finite elements for the case in which $m \le d+1$ are plotted in Table 11.1.

Global finite element spaces

We define our piecewise polynomial spaces W_h and V_h as follows. Again, V_h is a subspace of W_h in regard to the boundary conditions.

1. W_h consists of all functions v_h in $P_{m,h}$ such that for any $k \in \{1, \cdots, m\}$, any $(d-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any $\beta \in A_k$ with $|\beta| = m-k$, $d_{T,F,\beta}(v_h)$ is continuous through F.
2. V_h consists of all functions v_h in W_h such that for any $k \in \{1, \cdots, m\}$, any $(d-k)$-dimensional subsimplex F of any $T \in \mathcal{T}_h$ and any $\beta \in A_k$ with $|\beta| = m-k$, if $F \subset \partial\Omega$ then $d_{T,F,\beta}(v_h) = 0$.

For $v, w \in L^2(\Omega)$ that $v|_T, w|_T \in H^m(T)$, $\forall T \in \mathcal{T}_h$, we define

$$a_h(v,w) = \sum_{T\in\mathcal{T}_h}\int_T \Big(\sum_{|\alpha|=m} b_\alpha \partial^\alpha v\, \partial^\alpha w + b_0 v w\Big). \tag{11.17}$$

The nonconforming finite element method for problem (11.6) corresponding to is to find $u_h \in W_h$(or V_h, with respect to the boundary condition) such that

$$a_h(u_h, v_h) = (f, v_h), \quad \forall v_h \in W_h(\text{or } V_h, \text{ respectively}). \tag{11.18}$$

The convergent property of solution u_h of problem (11.18) will be discussed below.

11.4 Strang lemma and generalized patch test

11.4.1 Strang lemma

Let V be a Sobolev space, $a(\cdot,\cdot)$ is a continuous and coercive bilinear form on $V \times V$. We consider the variational problem: find $u \in V$, such that $a(u,v) = f(v)$ for all $v \in V$, where $f \in V'$.

Let V_h be a finite element space associated with V, equipped with $\|\cdot\|_h$. We consider the finite element problem: find $u_h \in V_h$, such that $a_h(u_h, v_h) = f_h(v_h)$, $\forall\, v_h \in V_h$, where $a_h(\cdot,\cdot)$ is a continuous and coercive bilinear form on $V_h \times V_h$ with respect to $\|\cdot\|_h$, and $f_h \in V_h'$. In practice, $a_h(\cdot,\cdot)$ is the bilinear form on $V_h \times V_h$ with numerical quadrature involved.

Define $A : V \to V'$ such that $(Aw, v) = a(w, v), \forall\, v \in V$, and $A_h : V_h \to V_h'$ such that $(A_h w_h, v_h) = a_h(w_h, v_h), \forall\, v_h \in V_h$. Then the boundary value problem equivalent to $Au = f$ in V', and the finite element method is equivalent to $A_h u_h = f_h$ in V_h'.

Denote $V^+ = V + V_h$. In practice, we assume there is a norm $\|\cdot\|_+$ equipped on V^+, such that $\|v\|_+ = \|v\|$ for $v \in V$ and $\|v_h\|_+ \overset{c}{\eqsim} \|v_h\|_h$ for $v_h \in V_h$. This way, u and u_h can lie in a same space, and we can estimate their distance. Moreover, let $a_+(\cdot,\cdot)$ be a bilinear form defined on $V_+ \times V_+$ which is continuous and coercive with respect to $\|\cdot\|_+$. Particularly when $V_h \subset V$, then $\|\cdot\|_+$ and $a(\cdot,\cdot)_+$ degenerate to $\|\cdot\|$ and $a(\cdot,\cdot)$, and in the typical nonconforming setting, $\|\cdot\|_+$ and $a_+(\cdot,\cdot)$ are the piecewise defined norm and bilinear form, respectively.

Define $\tilde{A} : V_+ \to V_h'$ such that $a_+(\xi, v_h) = (\tilde{A}\xi, v_h),\ \forall\, \xi \in V_+,\ v_h \in V_h$.

Theorem 82. *(Reformed Strang lemma) Let u and u_h be the solutions of the boundary value problem and the finite element problem, respectively. Let $\tilde{f} \in V_h'$ be induced by f.*

$$\|u - u_h\|_+ \lesssim \inf_{v_h \in V_h} \left(\|u - v_h\|_+ + \|\tilde{A}v_h - A_h v_h\|_{V_h'} \right) + \|\tilde{f} - \tilde{A}u\|_{V_h'} + \|f_h - \tilde{f}\|_{V_h'}. \tag{11.19}$$

Proof. Based on the coercivity of $a(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ with respect to $\|\cdot\|$ and $\|\cdot\|_h$, respectively, by a standard argument, we have $\|v_h\|_h \overset{c}{\eqsim} \|A_h v_h\|_{V_h'}$ for $v_h \in V_h$, and $\|v\| \overset{c}{\eqsim} \|Av\|_{V'}$ for $v \in V$.

Therefore, let $v_h \in V_h$, and we have

$$\begin{aligned}\|u_h - v_h\|_h \overset{c}{\eqsim} \|A_h(u_h - v_h)\|_{V_h'} &= \|A_h u_h - \tilde{f} + \tilde{f} - \tilde{A}u + \tilde{A}u - \tilde{A}v_h + \tilde{A}v_h - A_h u_h\|_{V_h'} \\ &\leqslant \|A_h u_h - \tilde{f}\|_{V_h'} + \|\tilde{f} - \tilde{A}u\|_{V_h'} + \|\tilde{A}u - \tilde{A}v_h\|_{V_h'} + \|\tilde{A}v_h - A_h v_h\|_{V_h'}.\end{aligned}$$

Note that by the continuity of $a_+(\cdot,\cdot)$, $\|\tilde{A}u - \tilde{A}v_h\|_{V_h'} \lesssim \|u - v_h\|_+$. Then,

$$\|u - u_h\|_+ \leqslant \|u - v_h\|_+ + \|v_h - u_h\|_+ \lesssim (\|u - v_h\|_+ + \|\tilde{A}v_h - A_h v_h\|_{V_h'}) + \|\tilde{f} - \tilde{A}u\|_{V_h'} + \|f_h - \tilde{f}\|_{V_h'}.$$

Since $v_h \in V_h$ is arbitrary, the theorem is proved by an infimum process. □

Remark 22.
1. $\|f_h - \tilde{f}\|_{V_h'} = \sup_{w_h \in V_h} \dfrac{(f - f_h)(w_h)}{\|w_h\|_h}$.
2. $\|\tilde{f} - \tilde{A}u\|_{V_h'} = \sup_{w_h \in V_h} \dfrac{f(w_h) - a_+(u, w_h)}{\|w_h\|_h}$
3. $\|\tilde{A}v_h - A_h v_h\|_{V_h'} = \sup_{w_h \in V_h} \dfrac{a_+(v_h, w_h) - a_h(v_h, w_h)}{\|w_h\|_h}$.

Remark 23. Comparison with the original form of the Strang Lemmas:

1. A unified form of the Strang Lemma is

$$\begin{aligned}\|u - u_h\|_+ \lesssim \inf_{v_h \in V_h} &\left(\|u - v_h\|_+ + \sup_{w_h \in V_h} \frac{a_+(v_h, w_h) - a_h(v_h, w_h)}{\|w_h\|_h} \right) \\ &+ \sup_{w_h \in V_h} \frac{f(w_h) - a_+(u, w_h)}{\|w_h\|_h} + \sup_{w_h \in V_h} \frac{(f - f_h)(w_h)}{\|w_h\|_h}.\end{aligned} \tag{11.20}$$

2. When $a_+(v_h, w_h) = a_h(v_h, w_h)$ for $v_h, w_h \in V_h$, then $\|\tilde{A}v_h - A_h v_h\|_{V_h'} = 0$. When further $f_h = \tilde{f}$ in V_h', the theorem degenerate to the second Strang lemma, namely,

$$\|u - u_h\|_+ \lesssim \inf_{v_h \in V_h} \|u - v_h\|_+ + \|\tilde{f} - \tilde{A}u\|_{V_h'} = \inf_{v_h \in V_h} \|u - v_h\|_+ + \sup_{w_h \in V_h} \frac{f(w_h) - a_+(u, w_h)}{\|w_h\|_h}.$$

When further $V_h \subset V$, the theorem degenerates to the Cea lemma.

3. When $V_h \subset V$, then $\|\tilde{f} - \tilde{A}u\|_{V_h'} = 0$. The theorem degenerates the first Strang lemma:

$$\|u - u_h\| \lesssim \inf_{v_h \in V_h} \left(\|u - v_h\| + \sup_{w_h \in V_h} \frac{a(v_h, w_h) - a_h(v_h, w_h)}{\|w_h\|_h} \right) + \sup_{w_h \in V_h} \frac{(f - f_h)(w_h)}{\|w_h\|_h}.$$

When further $a_h(\cdot, \cdot)$ coincides with $a(\cdot, \cdot)$ on $V_h \times V_h$, and f_h coincides with f on V_h, then the theorem degenerates to Cea lemma.

11.4.2 Generalised patch test

We will explain the connection between the general patch test and Lax equivalence theorem. We note here that the norm $\|\cdot\|_h$ is not necessarily be the nonconforming norm $\|\cdot\|_{m,h}$. The corresponding bilinear form a_h is assumed to satisfy the following condition.

Assumption 11.21 *The bilinear form a_h and energy norm $\|\cdot\|_h$ satisfies*

1. *Boundedness: $a_h(u_h, v_h) \lesssim \|u_h\|_h \|v_h\|_h$;*
2. *$a_h(u, v) = a(u, v)$, $\forall u, v \in V$;*
3. *Poincaré inequality: $\|v_h\|_{m,h} \lesssim \|v_h\|_h$.*

We first generalize the concepts of weak consistency and generalized patch test proposed for nonconforming finite element methods.

Definition 11 (Generalized patch test [?]). *For any $1 \le i \le n, |\alpha| < m$ and $\phi \in C_0^\infty(\Omega)$,*

$$\lim_{h \to 0} \sup_{\|v_h\|_h \le 1} \left| \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi \partial^\alpha v_h \nu_i \right| = 0. \tag{11.22}$$

Definition 12 (Weak consistency of V_h). *Given any bounded sequence in V_h that $\|v_h\|_h \le C$, if subsequence $\{v_{h_k}\}$ satisfies*

$$\partial_{h_k}^\alpha v_{h_k} \rightharpoonup v_\alpha \in L^2(\Omega) \quad k \to \infty,$$

then

$$\partial^\alpha v_0 = v_\alpha \quad \forall |\alpha| \le m. \tag{11.23}$$

Theorem 83. *The following statements are equivalent:*

1. *Generalized patch test passes;*
2. *Weak consistency holds;*
3. *It holds that*

$$\lim_{h \to 0} \sup_{v_h} \frac{a_h(u, v_h) - \langle f, v_h \rangle_h}{\|v_h\|_h} = 0. \tag{11.24}$$

Proof. $\boxed{1 \Longrightarrow 2}$ Assume $\partial_{h_k}^\alpha v_{h_k} \rightharpoonup v_\alpha$ and $\|v_{h_k}\| \le C$. For any $\phi \in C_0^\infty(\Omega)$, we have

$$\lim_{k \to \infty} \int_\Omega \partial_i \phi \partial_{h_k}^\alpha v_{h_k} = \int_\Omega \partial_i \phi v_\alpha, \qquad \lim_{k \to \infty} \int_\Omega \phi \partial_{h_k}^{\alpha + \boldsymbol{e}_i} v_{h_k} = \int_\Omega \phi v_{\alpha + \boldsymbol{e}_i}.$$

By GPT,

$$\begin{aligned} \int_\Omega \partial_i \phi v_\alpha + \phi v_{\alpha + \boldsymbol{e}_i} &= \lim_{k \to \infty} \int_\Omega (\partial_i \phi \partial_{h_k}^\alpha v_{h_k} + \phi \partial_{h_k}^{\alpha + \boldsymbol{e}_i} v_{h_k}) \\ &= \lim_{k \to \infty} \sum_{T \in \mathcal{T}_h} \int_{\partial T} \phi \partial^\alpha v_{h_k} \nu_i = 0, \end{aligned}$$

which gives $\partial^\alpha v_0 = v_\alpha$ by the definition of weak derivative.

$\boxed{2 \Longrightarrow 3}$ Let

$$\mu = \limsup_{h\to 0} \sup_{\|v_h\|_h=1} |\langle f, v_h\rangle_h - a_h(u, v_h)|.$$

Then, there exists $v_{h_k} \in V_{h_k}$, such that

$$\|v_{h_k}\|_h = 1, \qquad \mu = \lim_{k\to\infty} |\langle f, v_{h_k}\rangle_{h_k} - a_{h_k}(u, v_{h_k})|.$$

By Poincaré inequality, $\{\partial_{h_k}^\alpha\}$ is bounded in $L^2(\Omega)$, Then, there exists a subsequence (also denoted by v_{h_k}) such that $\partial_{h_k}^\alpha v_{h_k} \rightharpoonup v_\alpha$, which implies $\partial^\alpha v_0 = v_\alpha$ by the weak consistency. Therefore,

$$\begin{aligned}\lim_{k\to\infty}\langle f, v_{h_k}\rangle_{h_k} &= \lim_{k\to\infty}\sum_{|\alpha|\le m}(f_\alpha, \partial_{h_k}^\alpha v_{h_k}) \\ &= \sum_{|\alpha|\le m}(f_\alpha, v_\alpha) \\ &= \sum_{|\alpha|\le m}(f_\alpha, \partial^\alpha v_0) = \langle f, v_0\rangle.\end{aligned}$$

And,

$$\begin{aligned}\lim_{k\to\infty} a_{h_k}(u, v_{h_k}) &= \lim_{k\to\infty}\sum_{|\alpha|=m}(\partial^\alpha u, \partial_{h_k}^\alpha v_{h_k}) \\ &= \sum_{|\alpha|=m}(\partial^\alpha u, \partial^\alpha v_0) = a(u, v_0).\end{aligned}$$

Hence, $\mu = 0$.

$\boxed{3 \Longrightarrow 1}$ Recall that

$$\langle f, v_h\rangle_h := \sum_{|\alpha|\le m}(f_\alpha, \partial_h^\alpha v_h), \qquad f_\alpha \in L^2(\Omega).$$

Given $\phi \in C_0^\infty(\Omega)$, define

$$f_\beta = \begin{cases}\partial_i\phi & \beta = \alpha, \\ \phi & \beta = \alpha + \boldsymbol{e}_i, \\ 0 & \text{otherwise.}\end{cases}$$

It is straightforward that $u = 0$ is the solution of $(-\Delta)^m u = f$. Therefore, (11.24) implies

$$\lim_{h\to 0}\sup_{\|v_h\|_h=1} |\langle f, v_h\rangle_h| = 0,$$

Namely, for any $|\alpha| < m$

$$\lim_{h\to 0}\sup_{\|v_h\|_h=1}\left|\sum_{T\in\mathcal{T}_h}\int_T \partial_i\phi\partial^\alpha v_h + \phi\partial^{\alpha+\boldsymbol{e}_i} v_h\right| = 0,$$

which is GPT. □

By Lax Equivalence Theorem , under the stability condition, the convergence is equivalent to the consistency. The following theorem shows that Generalized patch test + approximability is equivalent to the consistency.

Theorem 84. *We have the following statement:*

Generalized patch test (or weak consistency) + approximability $\Longleftrightarrow$ Consistency.

Proof. "⟹": In light of definition of consistency and Theorem 83, by approximability and boundedness of a_h, we have

$$\begin{aligned} E_h^c(u) &= \inf_{w_h}\left(\|u-w_h\|_h + \sup_{v_h}\frac{a_h(w_h,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h}\right)\\ &\le \inf_{w_h}\left(\|u-w_h\|_h + \sup_{v_h}\frac{a_h(u,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h} + \sup_{v_h}\frac{a_h(w_h-u,v_h)}{\|v_h\|_h}\right)\\ &\lesssim \inf_{w_h}\|u-w_h\|_h + \sup_{v_h}\frac{a_h(u,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h} \to 0 \quad \text{as } h\to 0. \end{aligned}$$

"⟸": The consistency trivially implies the approximability. We also have for any w_h,

$$\begin{aligned} \sup_{v_h}\frac{a_h(u,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h} &= \sup_{v_h}\frac{a_h(u-w_h,v_h)+a_h(w_h,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h}\\ &\le \sup_{v_h}\frac{a_h(u-w_h,v_h)}{\|v_h\|_h} + \sup_{v_h}\frac{a_h(w_h,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h}\\ &\lesssim \|u-w_h\|_h + \sup_{v_h}\frac{a_h(w_h,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h}. \end{aligned}$$

Namely,

$$\sup_{v_h}\frac{a_h(u,v_h)-\langle f,v_h\rangle_h}{\|v_h\|_h} \lesssim E_h^c(u),$$

which gives (11.24), or equivalently, GPT. □

GPT for Morley

Let us take Morley element for example. The degrees of freedom on the edge implies that

$$\int_F [\frac{\partial u}{\partial \nu}] = 0.$$

For the tangential part, using integration by parts, we have

$$\int_F [\frac{\partial u}{\partial \tau}] = [u](a_i) - [u](a_j) = 0 \qquad \tau = \overrightarrow{a_j a_i}.$$

Let $P_e^0 : L^2(e) \mapsto \mathcal{P}_0(e)$ be the orthogonal projection. We will verify the GPT (11.22) in the following two cases:

1. $|\alpha| = 1$. For any $1 \le i \le n$ and $\phi \in C_0^\infty(\Omega)$

$$\begin{aligned} \sum_{T\subset\mathcal{T}_h}\int_{\partial T}\phi\partial^\alpha v_h \nu_i &= \sum_{e\subset\mathcal{E}_h}\int_e \phi[\partial^\alpha v_h]\nu_i\\ &= \sum_{e\subset\mathcal{E}_h}\int_e (\phi - P_e^0\phi)[\partial^\alpha v_h - P_e^0\partial^\alpha v_h]\nu_i\\ &\lesssim h\|\phi\|_1\|v_h\|_{2,h} \to 0 \quad \text{as} \quad h\to 0. \end{aligned}$$

2. $|\alpha| = 0$. For any $\phi \in C_0^\infty(\Omega)$, we have

$$\sum_{T\subset\mathcal{T}_h}\int_{\partial T}\phi v_h = \sum_{T\subset\mathcal{T}_h}\int_{\partial T}\phi(v_h - v_0) \lesssim h\|\phi\|_1(h^{-1}|v_h - v_0|_{0,h} + |v_h - v_0|_{1,h}) \lesssim h\|\phi\|_1\|v_h\|_{2,h}.$$

Here, v_0 can be chosen as the interpolation to $\mathcal{P}_1$ or $\mathcal{P}_2$ Lagrange space.

Hence, Morley element is convergent element for the 4th-order problem.

11.5 Error estimate of the MW^{2}X element

Lemma 85. *If $F \not\subseteq \partial\Omega$ or $v_h \in V_{h0}$, then*

$$[\partial_h^\alpha v_h]_F^2(x) \lesssim h^{2(m-|\alpha|)-n}\sum_{T\in\mathcal{T}_h, F\subset T}|v_h|_{m,T}^2, \quad \forall x \in F.$$

Proof. By the weak continuity or the weak zero-boundary condition, there exist $y \in F$ such that $[\partial_h^\alpha v_h]_F$ vanishes at y, this leads to

$$[\partial_h^\alpha v_h]_F^2(x) = (\int_{\mathbf{yx}}[\frac{\partial}{\partial\tau}\partial_h^\alpha v_h]_F(z)dz)^2 \lesssim h^2\max_{z\in F}[\frac{\partial}{\partial\tau}\partial_h^\alpha v_h]_F^2(z) \lesssim h^2\sum_{|\beta|=|\alpha|+1}\max_{z\in F}[\partial_h^\beta v_h]_F^2(z),$$

where τ is the direction of the vector $\mathbf{yx}$. Repeating the same argument, we have

$$[\partial_h^\alpha v_h]_F^2(x) \lesssim h^{2(m-|\alpha|)}\sum_{|\beta|=m}\max_{z\in F}[\partial_h^\beta v_h]_F^2(z).$$

By the inverse inequality, we finish this proof. □

Theorem 85. *Let $u \in H_0^m(\Omega)$ be the solution of the $2m - th$ order equation, and let $u_h \in V_{h0}$ be the solution of the discrete problem for WMX element space. Then for any $f \in L^2(\Omega)$,*

$$\|u - u_h\|_{m,h} = \sum_{k=1}^{m-1}h^k|u|_{m+k} + h^m\|f\|,$$

when $u \in H^{2m}(\Omega)$.

Proof. By Strong lemma, we should only consider the next term

$$a_h(u, v_h) - (f, v_h) = \sum_{T\in\mathcal{T}_h}\sum_{|\alpha|=m}(\partial^\alpha u, \partial^\alpha v_h)_T - (f, v_h)$$

with $f = \sum_{|\alpha|=m}(-1)^m\partial^\alpha\partial^\alpha u$. So without loss of generality, for a given α with $|\alpha| = m$ it can be written as $\alpha = e_{\alpha_1} + e_{\alpha_2} + \cdots + e_{\alpha_m}$, and we note that:

$$\alpha(k) = \sum_{i=1}^k e_{\alpha_i}, \quad \alpha^{'}(k) = \alpha - \alpha(k), \quad 0 \le k \le m.$$

By using of integral by part, we have that:

$$a_h(u,v_h)-(f,v_h)=\begin{cases}E_1, & m=1,\\ E_1+E_2, & m>1.\end{cases}$$

with

$$E_1=\sum_{|\alpha|=m}\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F \partial^\alpha u\partial^{\alpha'(1)}v_h\nu_{\alpha(1)}ds,$$

and

$$E_2=\sum_{|\alpha|=m}\sum_{k=1}^{m-1}(-1)^k\sum_{F\in\mathcal{E}_h}\int_F \partial^{\alpha+\alpha(k)}u[\partial^{\alpha'(k+1)}v_h]\nu_{\alpha_{k+1}}ds,$$

Now we can analyse E_1 and E_2 one by one:

- For E_1, we can use the continuity of $[\partial^{\alpha'(1)}v_h]$ across the face F. So we have:

$$\begin{aligned}\sum_{|\alpha|=m}\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F\left(\partial^\alpha u\partial^{\alpha'(1)}v_h\right)ds &\lesssim \sum_{|\alpha|=m}\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\int_F\left(\partial^\alpha u-P_F^0\partial^\alpha u\right)\left(\partial^{\alpha'(1)}v_h-P_F^0\partial^{\alpha'(1)}v_h\right)ds\\ &\lesssim \sum_{|\alpha|=m}\sum_{T\in\mathcal{T}_h}\sum_{F\subset\partial T}\|\partial^\alpha u-P_F^0\partial^\alpha u\|_{0,F}\|\partial^{\alpha'(1)}v_h-P_F^0\partial^{\alpha'(1)}v_h\|_{0,F}\\ &\lesssim h\sum_{T\in\mathcal{T}}|u|_{m+1,T}|v_h|_{m,h,T}\lesssim h|u|_{m+1}|v_h|_{m,h}.\end{aligned}$$

- If E_2, we can use the trace inequality with scaling and the lemma above. For given $k=1,2,\cdots,m-1$,

$$\begin{aligned}&\sum_{|\alpha|=m}(-1)^k\sum_{F\in\mathcal{E}_h}\int_F\partial^{\alpha+\alpha(k)}u[\partial^{\alpha'(k+1)}v_h]\nu_{\alpha_{k+1}}ds\\ \lesssim&\sum_{|\alpha|=m}\left(\sum_{F\in\mathcal{E}_h}\|\partial^{\alpha+\alpha(k)}u\|^2\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{E}_h}\|[\partial^{\alpha'(k+1)}v_h]\|^2\right)^{\frac{1}{2}}\\ \lesssim&\left(\sum_{F\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_h(F)}h^{-1}|u|_{m+k,T'}^2+h|u|_{m+k+1,T'}^2\right)^{\frac{1}{2}}\left(\sum_{F\in\mathcal{E}_h}\sum_{T\in\mathcal{T}_h(F)}h^{n-1}\cdot h^{2(k+1)-n}|v_h|_{m,T'}^2\right)^{\frac{1}{2}}\\ \lesssim& h^k|u|_{m+k}|v_h|_{m,h}+h^{k+1}|u|_{m+k+1}|v_h|_{m,h}.\end{aligned}$$

  So we have:

$$E_2\lesssim\sum_{k=1}^m h^k|u|_{m+k}|v_h|_{m,h}.$$

  At last, we have that

$$E_1+E_2\lesssim\sum_{k=1}^m h^k|u|_{m+k}|v_h|_{m,h},$$

which finishes this proof. □

11.6 Interior penalty nonconforming finite element methods ($m, n\geq 1$)

In this section, we will present a family of nonconforming finite element methods that converges for arbitrary $m,n\geq 1$ [**?**].

11.6.1 Definition of nonconforming finite elements

For any $m \geq 0$ and $d \geq 1$, we define $L = \lfloor \frac{m}{d+1} \rfloor$, where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x. A finite element can be represented by a triple (T, P_T, D_T), where T is the geometric shape of the element, P_T is the shape function space, and D_T is the set of the degrees of freedom that is P_T-unisolvent. The minimal shape function space is defined as $P_T^{(m,d)} := \mathcal{P}_m(T)$, where $\mathcal{P}_k(T)$ denotes the space of all polynomials defined on T with a degree not greater than k, for any integer $k \geq 0$.

For $0 \leq k \leq d$, let $\mathcal{F}_{T,k}$ be the set consisting of all $(n-k)$-dimensional sub-simplexes of T (Hence, k represents the co-dimension). For any $F \in \mathcal{F}_{T,k}$ $(1 \leq k \leq n)$, let $|F|$ denote its $(n-k)$-dimensional measure, and $\nu_{F,1}, \cdots, \nu_{F,k}$ be linearly independent unit vectors that are orthogonal to the tangent space of F. Specifically, F represents a vertex and $|F| = 1$ when $k = d$.

For $1 \leq k \leq n$, let A_k be the set consisting of all multi-indexes α with $\sum_{i=k+1}^{d} \alpha_i = 0$. For any $(d-k)$-dimensional sub-simplex $F \in \mathcal{F}_{T,k}$ and $\alpha \in A_k$ with

$$|\alpha| = m - k - (d+1)(L-l) \qquad 0 \leq l \leq L,$$

define

$$\begin{aligned} d_{T,F,\alpha}(v) &:= \frac{1}{|F|}\int_F \frac{\partial^{|\alpha|} v}{\partial \nu_{F,1}^{\alpha_1} \cdots \partial \nu_{F,k}^{\alpha_k}} & \forall v \in H^m(T), \\ d_{T,0}(v) &:= \frac{1}{|T|}\int_T v & \forall v \in H^m(T). \end{aligned} \tag{11.25}$$

Here, l represents the level of the degrees of freedom. By the Sobolev embedding theorem (cf. [?]), $d_{T,F,\alpha}$ is continuous linear functional on $H^m(T)$. We define the set of the degrees of freedom at level l as

$$\begin{aligned} \tilde{D}_{T,l}^{(m,d)} := \Big\{ d_{T,F,\alpha} \mid \alpha \in A_k \text{ with } |\alpha| = m - k - (d+1)(L-l), \\ F \in \mathcal{F}_{T,k}, 1 \leq k \leq \min\{d, m-(d+1)(L-l)\} \Big\} \qquad \text{for } 0 \leq l \leq L. \end{aligned} \tag{11.26}$$

We further define the set of degrees of freedom at level $l = -1$ as

$$\tilde{D}_{T,-1}^{(m,d)} := \begin{cases} \{d_{T,0}\} & \text{if } m \equiv 0 \ (\text{mod } d+1), \\ \varnothing & \text{otherwise.} \end{cases} \tag{11.27}$$

Then, the set of the degrees of freedom is

$$D_T^{(m,d)} = \bigcup_{l=-1}^{L} \tilde{D}_{T,l}^{(m,d)}. \tag{11.28}$$

The diagrams of the finite elements $(0 \leq m \leq 5, 1 \leq d \leq 3)$ are plotted in Table 11.2.

Remark 24. It can be seen that if $m \leq d$, namely $L = 0$, the finite element is exactly the Morly-Wang-Xu element proposed in [?].

Remark 25. The set of degrees of freedom lacks the derivatives of order $m - (d+1)(L-l+1)$ on the sub-simplexes for any $1 \leq l \leq L$. Hence, we impose the interior penalty terms on the $(m-(d+1)(L-l+1))$-th order derivatives when defining the bilinear form.

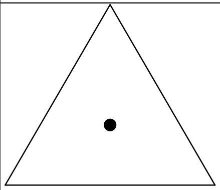
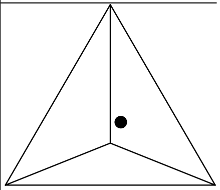
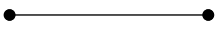
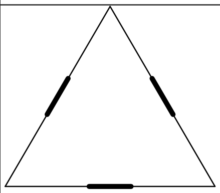
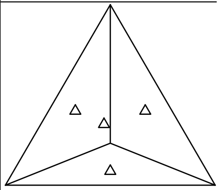
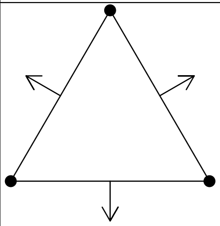
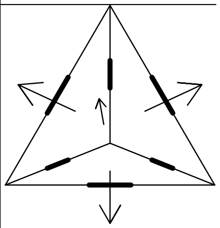
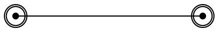
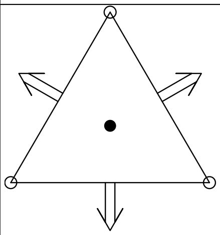
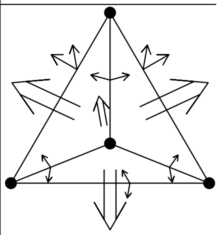
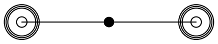
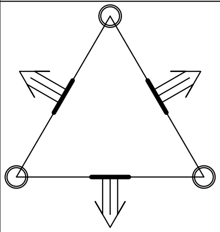
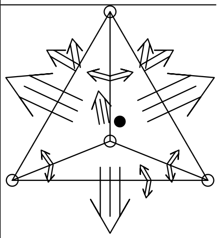
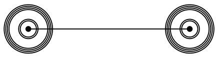
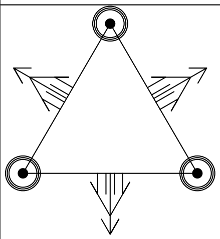
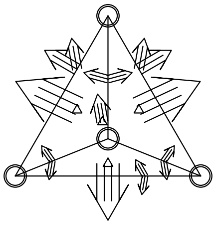
$m\backslash d$	1	2	3
0			
1			
2			
3			
4			
5			

Table 11.2. Degrees of freedom: $0 \leq m \leq 5$, $1 \leq d \leq 3$

11.6.2 Interior penalty nonconforming methods

We immediately know that $V_h^{(m,d)}$ (resp. $V_{h0}^{(m,d)}$) satisfies the k-weak continuity (resp. k-weak zero-boundary condition) if $k \neq m-(d+1)(L-l+1)$ $(1 \leq l \leq L)$. That is, the nonconforming finite element spaces do not satisfy the weak continuity or weak zero-boundary condition in general when $L \geq 1$. We, therefore, introduce the interior penalty as a remedy. We denote $V_h = V_{h0}^{(m,d)}$ as the nonconforming approximation of $H_0^m(\Omega)$. For any $w, v \in V_h + H_0^m(\Omega)$, consider the following bilinear form

$$\begin{aligned} a_h(w,v) := & \sum_{|\alpha|=m} (\partial_h^\alpha w, \partial_h^\alpha v) \\ & + \eta \sum_{l=1}^{L} \sum_{F \in \mathcal{F}_h} h_F^{1-2(n+1)(L-l+1)} \int_F \sum_{|\beta|=m-(d+1)(L-l+1)} [\![\partial_h^\beta w]\!] \cdot [\![\partial_h^\beta v]\!], \end{aligned} \tag{11.29}$$

where $\eta = O(1)$ is a given positive constant. Then, the interior penalty nonconforming finite element methods for problem (**??**) read: Find $u_h \in V_h$, such that

$$a_h(u_h, v_h) = (f, v_h) \qquad \forall v_h \in V_h. \tag{11.30}$$

Then, we define

$$\|v\|_h^2 := |v|_{m,h}^2 + \sum_{l=1}^{L} \sum_{F \in \mathcal{F}_h} h_F^{1-2(d+1)(L-l+1)} \sum_{|\beta|=m-(d+1)(L-l+1)} \|[\![\partial^\beta v]\!]\|_{0,F}^2 \qquad \forall v \in V_h + H_0^m(\Omega), \tag{11.31}$$

which can be proved a norm on $V_h + H_0^m(\Omega)$ by generalized Poincaré inequality.

Remark 26. For the case in which $m \leq d$, we have $L = \lfloor \frac{m}{d+1} \rfloor = 0$. Then, $a_h(\cdot,\cdot)$ and $\|\cdot\|_h$ are exactly the bilinear form and norm for the nonconforming finite element method, respectively.

GPT when $d = 2, m = 3$

The proposed IP nonconforming finite element passes the GPT. We will take the case in which $d = 2, m = 3$ as an illustration, where the variational formulation reads:

$$(\nabla_h^3 u_h, \nabla_h^3 v_h) + \eta \sum_{e \in \mathcal{E}_h} h_e^{-5} \int_e [\![u_h]\!] \cdot [\![v_h]\!] = \langle f, v_h \rangle_h \qquad \forall v_h \in V_h,$$

The corresponding energy norm is defined as

$$\|v_h\|_h^2 := |v_h|_{3,h}^2 + \|h_e^{-5/2} [\![v_h]\!]\|_{0,\mathcal{E}_h}^2.$$

The GPT for $|\alpha| = 2$ and $|\alpha| = 1$ is similar to the nonconforming methods (HW). For $|\alpha| = 0$, we have $\|[\![v_h]\!]\|_{0,\mathcal{E}_h} \eqsim h^{5/2} \to 0$ for all $\|v_h\|_h = 1$. Then,

$$T_{\alpha,i}(\phi, v_h) = \sum_{e \in \mathcal{E}_h} \int_e \phi [v_h] v_i ds \to 0 \quad h \to 0.$$

which shows that the IP nonconforming method is convergent for all $f \in H^{-m}(\Omega)$.