

## Finite Volume Methods

### 17.1 The finite volume method

#### 17.1.1 Triangulations and basis functions

Let  $\mathcal{T}_h$  be a triangulation of  $\Omega$ . Let  $x_i$ ,  $1 \leq i \leq N$ , denote the vertices of the triangulation. With each vertex  $x_i$ , we associate a region  $\Omega_i$  consisting of those triangles  $\tau \in \mathcal{T}_h$  which have  $x_i$  as a vertex.

We now construct a dual mesh  $\mathcal{T}_h^*$  of  $\mathcal{T}_h$ . The elements in the dual mesh, sometimes called boxes, are constructed as follows: For each triangle  $\tau$ , select a distinguished point  $p \in \tau$ . Connect  $p$  by straight-line segments ( $e_1, e_2, e_3$  in Fig.17.4) to the edge midpoints of  $\tau$  ( $m_1, m_2, m_3$  in Fig.17.4). This partitions  $\tau$  into three subregions (with areas  $a_1, a_2, a_3$ ).

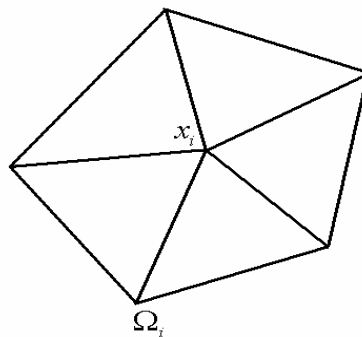


Fig. 17.1.  $\Omega_i$

With each vertex  $x_i$ , we will associate the box  $b_i \in \mathcal{T}_h^*$ ,  $b_i \subset \Omega_i$ , which consists of the union of the subregions in  $\Omega_i$  which have  $x_i$  as a corner (see Fig.17.5). Let  $V_h \subset H^1(\Omega)$  be the standard piecewise linear finite element space with respect to  $\mathcal{T}_h$  and  $W_h$  be the piecewise constant space with respect to  $\mathcal{T}_h^*$ . Let  $\phi_i$  denote the usual nodal basis function for  $V_h$ , satisfying

$$\phi_i(x_j) = \delta_{ij}$$

and  $\bar{\phi}_i$  denote the basis for  $W_h$  consisting of the characteristic functions for  $b_i$ :

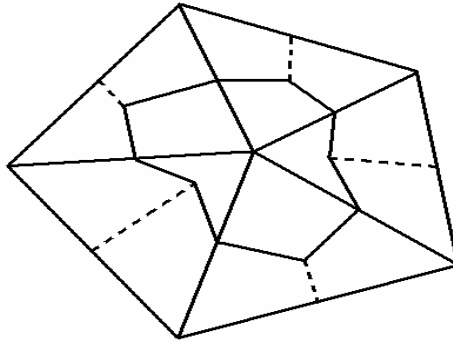


Fig. 17.2. A finite volume box

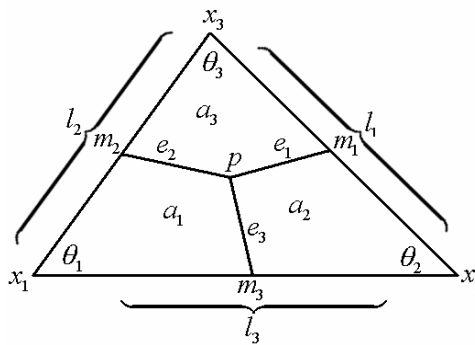


Fig. 17.3. A finite volume element

$$\bar{\phi}_i(x) = \begin{cases} 1 & x \in b_i \\ 0 & \text{otherwise} \end{cases}$$

Let  $v \in V_h$  be expressed in terms of  $\phi_i$  as

$$v = \sum_i v_i \phi_i(x)$$

we associate  $v$  with the element  $\bar{v} \in W_h$  defined by

$$G(v) \equiv \bar{v} = \sum_i v_i \bar{\phi}_i(x)$$

### 17.1.2 The finite volume method

Consider the following model problem:

$$-\nabla(\rho \nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The finite volume element method can be formulated as follows: Find  $u_h \in V_h$  such that

$$\bar{a}(u_h, v) = (f, \bar{v}) \quad \forall \bar{v} \in W_h$$

where

$$\bar{a}(u_h, v) = - \sum_{b \in \mathcal{T}_h^*} \int_{\partial b} \rho \frac{\partial u}{\partial n} v.$$

Equivalently

$$- \int_{\partial b} \rho \frac{\partial u_h}{\partial n} v = \int_b f \quad \forall b \in \mathcal{T}_h^*.$$

### 17.1.3 Relationship between finite volume and finite element method

For the Poisson equation, the finite volume method is not much different from the finite element method. In fact, we can prove that the stiffness matrices for these two difference methods are identical. Hence these two methods only differ in their right hand side. But for problems with variable coefficients, the difference can be more significant.

**Lemma 120.** Let  $u, v \in V_h$  and  $\bar{v} = G(v) \in W_h$ , then

$$(17.1) \quad \int_{\Omega} \nabla u \cdot \nabla v = - \sum_{b \in \mathcal{T}_h^*} \int_{\partial b} \frac{\partial u}{\partial n} \bar{v}$$

*Proof.* By Green's formula,

$$- \int_{\partial b_i \cap \tau} \frac{\partial u}{\partial n} = \int_{b_i \cap \partial \tau} \frac{\partial u}{\partial n} = \int_{\partial \tau} \frac{\partial u}{\partial n} \phi_i = \int_{\tau} \nabla u \cdot \nabla \phi_i.$$

The desired result then follows.  $\square$

### 17.1.4 Triangulations and basis functions

Let  $\mathcal{T}_h$  be a triangulation of  $\Omega$ . Let  $x_i, 1 \leq i \leq N$ , denote the vertices of the triangulation. With each vertex  $x_i$ , we associate a region  $\Omega_i$  consisting of those triangles  $\tau \in \mathcal{T}_h$  which have  $x_i$  as a vertex.

We now construct a dual mesh  $\mathcal{T}_h^*$  of  $\mathcal{T}_h$ . The elements in the dual mesh, sometimes called boxes, are constructed as follows: For each triangle  $\tau$ , select a distinguished point  $p \in \tau$ . Connect  $p$  by straight-line segments ( $e_1, e_2, e_3$  in Fig.17.4) to the edge midpoints of  $\tau$  ( $m_1, m_2, m_3$  in Fig.17.4). This partitions  $\tau$  into three subregions (with areas  $a_1, a_2, a_3$ ).

With each vertex  $x_i$ , we will associate the box  $b_i \in \mathcal{T}_h^*$ ,  $b_i \subset \Omega_i$ , which consists of the union of the subregions in  $\Omega_i$  which have  $x_i$  as a corner (see Fig.17.5). Let  $V_h \subset H^1(\Omega)$  be the standard piecewise linear finite element space with respect to  $\mathcal{T}_h$  and  $W_h$  be the piecewise constant space with respect to  $\mathcal{T}_h^*$ . Let  $\phi_i$  denote the usual nodal basis function for  $V_h$ , satisfying

$$\phi_i(x_j) = \delta_{ij}$$

and  $\bar{\phi}_i$  denote the basis for  $W_h$  consisting of the characteristic functions for  $b_i$ :

$$\bar{\phi}_i(x) = \begin{cases} 1 & x \in b_i \\ 0 & \text{otherwise} \end{cases}$$

Let  $v \in V_h$  be expressed in terms of  $\phi_i$  as

$$v = \sum_i v_i \phi_i(x)$$

we associate  $v$  with the element  $\bar{v} \in W_h$  defined by

$$G(v) \equiv \bar{v} = \sum_i v_i \bar{\phi}_i(x)$$

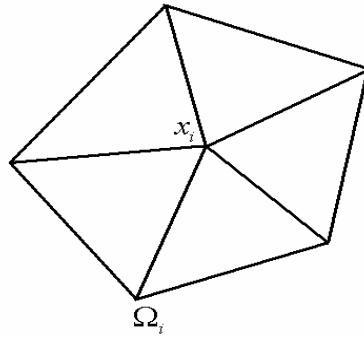


Fig. 17.4.  $\Omega_i$

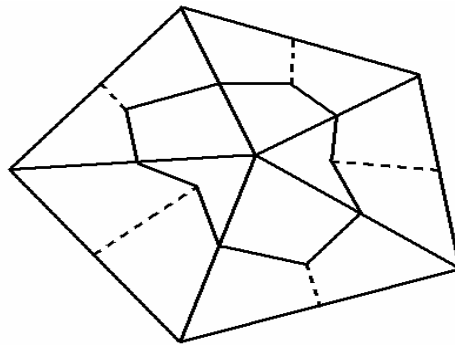


Fig. 17.5. A finite volume box

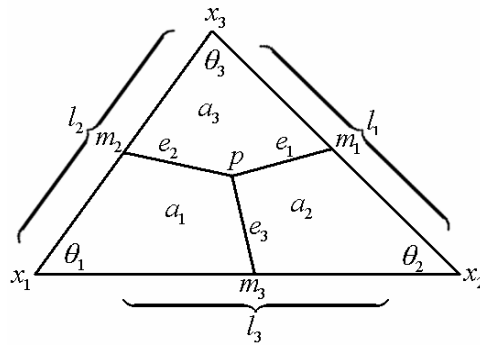


Fig. 17.6. A finite volume element

### 17.1.5 Delaunay triangulations

**Theorem:** Energy minimizing property

Given all the grid points, then the Delaunay triangulation gives the minimal energy (Dirichlet integral) for all the finite element functions having the same boundary values.

### 17.1.6 The finite volume element method

Consider the following model problem:

$$-\nabla(\rho\nabla u) = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega.$$

The finite volume element method can be formulated as follows: Find  $u_h \in V_h$  such that

$$\bar{a}(u_h, v) = (f, \bar{v}) \quad \forall \bar{v} \in W_h$$

where

$$\bar{a}(u_h, v) = - \sum_{b \in \mathcal{T}_h^*} \int_{\partial b} \rho \frac{\partial u}{\partial n} v.$$

Equivalently

$$- \int_{\partial b} \rho \frac{\partial u_h}{\partial n} v = \int_b f \quad \forall b \in \mathcal{T}_h^*.$$

### 17.1.7 Error analysis

We should carry out an analysis for  $\rho = 1$  and the more general case is left as an exercise.

**Lemma 121.** Let  $v \in V_h, \bar{v} \in W_h$ . Then there exists a constant

$$|v|_{1,\Omega}^2 \approx \sum_{e \in E} |[\bar{v}]|^2$$

and

$$\|v\|_{0,\Omega} \approx \|\bar{v}\|_{0,\Omega}$$

where  $E$  is the set of all edges in  $\mathcal{T}_h^*$  and  $[\bar{v}]$  denotes the jump of  $\bar{v}$  on  $e$ .

*Proof.* The proof follows from the following identities

$$|v|_{1,\tau}^2 = d_1(u_2 - u_3)^2 + d_2(u_3 - u_1)^2 + d_3(u_1 - u_2)^2$$

with  $d_i = l_i l_j \cos \theta_i / (4|\tau|)$ , and

$$\sum_{e \in E \cap \tau} |[\bar{v}]|^2 = (u_2 - u_3)^2 + (u_3 - u_1)^2 + (u_1 - u_2)^2,$$

$$\|v\|_{0,\Omega}^2 = \frac{|\tau|}{12} (u_1^2 + u_2^2 + u_3^2 + (u_1 + u_2 + u_3)^2)$$

and

$$\|v\|_{0,\Omega}^2 = a_1 u_1^2 + a_2 u_2^2 + a_3 u_3^2$$

□

**Lemma 122.** Let  $u, v \in V_h$  and  $\bar{v} = G(v) \in W_h$ , then

$$(17.2) \quad \int_{\Omega} \nabla u \cdot \nabla v = - \sum_{b \in \mathcal{T}_h^*} \int_{\partial b} \frac{\partial u}{\partial n} \bar{v}$$

*Proof.* It suffices to prove (17.4) for the basis function  $v = \phi_i$  and in this case (17.4) reduces to

$$(17.3) \quad \int_{\Omega_i} \nabla u \cdot \nabla \phi_i = - \int_{\partial b_i} \frac{\partial u}{\partial n}$$

Let  $t = [x_1, x_2, x_3] \subset \Omega_i$ .

$$\int_{\tau} \nabla u \cdot \nabla \phi_i = (1, 0, 0) A_{\tau} (u_1, u_2, u_3)^t = d_2(u_1 - u_3) + d_3(u_1 - u_2)$$

Next, note that since  $u$  is a linear polynomial in  $\tau$ ,  $\Delta u = 0$  in  $t$ , and by Green's theorem, the integral

$$\int_{\partial b_i \cap \tau} \frac{\partial u}{\partial n}$$

depends only on the endpoints of the integration path (in this case, midpoints  $m_2$  and  $m_3$  in Fig) and not on the location of  $p$ . One can thus choose a simple integration path (say, the perpendicular bisectors of sides 2 and 3 of  $\tau$ ) and directly compute

$$- \int_{\partial b_i \cap \tau} \frac{\partial u}{\partial n} = d_2(u_1 - u_3) + d_3(u_1 - u_2) = \int_{\tau} \nabla u \cdot \nabla \phi_i.$$

□

**Another proof of the lemma:** By Green's formula,

$$0 = \int_{b_i \cap \tau} \nabla u \cdot \nabla 1 = \int_{\partial(b_i \cap \tau)} \frac{\partial u}{\partial n} = \int_{\partial b_i \cap \tau} \frac{\partial u}{\partial n} + \int_{b_i \cap \partial \tau} \frac{\partial u}{\partial n}.$$

Thus

$$- \int_{\partial b_i \cap \tau} \frac{\partial u}{\partial n} = \int_{b_i \cap \partial \tau} \frac{\partial u}{\partial n} = \int_{\partial \tau} \frac{\partial u}{\partial n} \phi_i = \int_{\tau} \nabla u \cdot \nabla \phi_i.$$

**Lemma 123 (inf-sup condition).**

$$\inf_{v_h \in V_h} \sup_{\bar{\phi}_h \in W_h} \frac{\bar{a}(v_h, \bar{\phi}_h)}{|v_h|_{1,\Omega} \|\bar{\phi}_h\|} \geq 1$$

where  $\|\bar{\phi}_h\| = |\phi_h|_{1,\Omega}$  with  $\phi_h = G^{-1}(\bar{\phi}_h)$ .

*Proof.* By Lemma 122

$$\sup_{\bar{\phi}_h \in W_h} \frac{\bar{a}(v_h, \bar{\phi}_h)}{|v_h|_{1,\Omega}} \geq \sup_{\phi_h \in V_h} \frac{a(v_h, \phi_h)}{|v_h|_{1,\Omega}} = |\phi_h|_{1,\Omega} = \|\bar{\phi}_h\|$$

□

**Theorem 119.** Let  $u_h$  be the finite volume element solution. Then

$$|u - u_h|_{1,\Omega} \lesssim h|u|_{2,\Omega}.$$

*Proof.* Let  $u_I \in V_h$  be the linear interpolation of  $u$ .

$$\bar{a}(u - u_I, \bar{v}) = \sum_e [\bar{v}] \int_e \frac{\partial(u - u_I)}{\partial n} \lesssim \left( \sum_e |e| \int_e \left( \frac{\partial(u - u_I)}{\partial n} \right)^2 \right)^{1/2}$$

We consider the last term above on a triangle by triangle basis. Note that  $|e_i| \lesssim h$ , if  $e_i \tau$ . Using standard trace inequality and noting that  $\Delta u_I = 0$  on  $\tau$ , we have

$$\begin{aligned} \sum_{i=1}^3 h_\tau \int_{e_i} \left( \frac{\partial(u - u_I)}{\partial n} \right)^2 &\lesssim h_\tau \left( h_\tau \|\Delta(u - u_I)\|_{0,\tau}^2 + h_\tau^{-1} \|\nabla(u - u_I)\|_{0,\tau}^2 \right) \\ &\lesssim h_\tau^2 |u|_{2,\tau}^2. \end{aligned}$$

Using the following identity

$$\bar{a}(u_h - u_I, \bar{\phi}_h) = \bar{a}(u - u_I, \bar{\phi}_h)$$

and Lemma 123, we obtain

$$|u_h - u_I|_{1,\Omega} \lesssim h |u|_{2,\Omega}.$$

The desired result then follows by combining a standard estimate for  $u - u_I$ .  $\square$

**17.1.8 Relationship between finite volume and finite element method**

For the Poisson equation, the finite volume method is not much different from the finite element method. In fact, we can prove that the stiffness matrices for these two difference methods are identical. Hence these two methods only differ in their right hand side. But for problems with variable coefficients, the difference can be more significant.

**Lemma 124.** *Let  $u, v \in V_h$  and  $\bar{v} = G(v) \in W_h$ , then*

$$(17.4) \quad \int_{\Omega} \nabla u \cdot \nabla v = - \sum_{b \in \mathcal{T}_h^*} \int_{\partial b} \frac{\partial u}{\partial n} \bar{v}$$

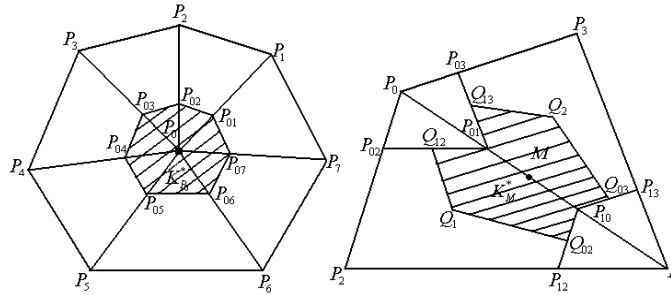
*Proof.* By Green's formula,

$$- \int_{\partial b_i \cap \tau} \frac{\partial u}{\partial n} = \int_{b_i \cap \partial \tau} \frac{\partial u}{\partial n} = \int_{\partial \tau} \frac{\partial u}{\partial n} \phi_i = \int_{\tau} \nabla u \cdot \nabla \phi_i.$$

The desired result then follows.  $\square$

**17.1.9 Higher order finite volume methods**

The finite volume method can be extended to higher orders. Let us now quadratic element as the trial space. In this case, the degree freedoms are on all vertices plus the mid points of the edges. We then need to construct control volumes associated with all these degree freedoms. For each vertex, the control volume is shown in the left picture in Fig.17.7 where  $P_{0i}$  is one-third location on  $P_0P_i$ ; for each edge midpoint, the control volume is shown in the right picture in Fig.??.



**Fig. 17.7.** Dual meshes for quadratic finite volume elements

It can be proved that the corresponding finite volume element solution  $u_h$  admits the following error estimate

$$\|u - u_h\|_{1,\Omega} \lesssim h^2 |u|_{3,\Omega}.$$