14

Mixed Methods for Stokes equations

14.1 Navier-stokes Equations

Given the domain $\Omega \subset \mathbb{R}^n$, we consider function $\boldsymbol{u} : \Omega \mapsto \mathbb{R}^m$, $(m = 1, n)$. Depending on m, $\boldsymbol{u}$ can be scalar-valued or vector-valued. In dynamical setting, we consider time dependent function $\boldsymbol{u} : \Omega \times (0, T) \mapsto \mathbb{R}^n$.

Vector and Matrix Derivatives

Derivative Notation

Let $m = n$. We write $\boldsymbol{u}$ as a column vector

$$\boldsymbol{u}(\hat{\boldsymbol{x}}) = \begin{bmatrix} u_1(\hat{\boldsymbol{x}}) \\ \vdots \\ u_n(\hat{\boldsymbol{x}}) \end{bmatrix}.$$

Recall the Fréchet derivative

$$D_{\boldsymbol{x}} \boldsymbol{u}(\xi) = \frac{d}{dt} \boldsymbol{u}(\boldsymbol{x} + t\xi)|_{t=0}$$

and

$$\frac{d}{dt} \boldsymbol{u}(\boldsymbol{x} + t\xi) = \frac{d}{dt} \begin{pmatrix} u_1(\boldsymbol{x} + t\xi) \\ \vdots \\ u_m(\boldsymbol{x} + t\xi) \end{pmatrix} = \begin{pmatrix} \nabla u_1 \cdot \xi \\ \vdots \\ \nabla u_m \cdot \xi \end{pmatrix} = \begin{pmatrix} (\nabla u_1)^T \\ \vdots \\ (\nabla u_m)^T \end{pmatrix} \xi.$$

In order to be consistent with Fréchet derivative, we write the derivatives with respect to $\hat{\boldsymbol{x}} = [X_1 \cdots X_m]^T$ as

$$\frac{\partial \boldsymbol{u}}{\partial \hat{\boldsymbol{x}}}(\hat{\boldsymbol{x}}) = \begin{bmatrix} (\nabla u_1(\hat{\boldsymbol{x}}))^T \\ \vdots \\ (\nabla u_n(\hat{\boldsymbol{x}}))^T \end{bmatrix} = \left[\frac{\partial u_i}{\partial X_j} \right]_{ij}.$$

We also use the gradient notation

$$\nabla_{\hat{\boldsymbol{x}}} \boldsymbol{u} = \frac{\partial \boldsymbol{u}}{\partial \hat{\boldsymbol{x}}}$$

interchangeably to denote the derivative. Divergence of a vector function $\boldsymbol{u}$ is defined as following:

$$\mathrm{div}_{\hat{\boldsymbol{x}}}\boldsymbol{u} = \nabla_{\hat{\boldsymbol{x}}} \cdot \boldsymbol{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial X_i}$$

Given a matrix function

$$A(\hat{\boldsymbol{x}}) = \begin{bmatrix} \boldsymbol{a}_1^T(\hat{\boldsymbol{x}}) \\ \vdots \\ \boldsymbol{a}_n^T(\hat{\boldsymbol{x}}) \end{bmatrix}$$

The divergence of a matrix $A(\hat{\boldsymbol{x}})$ is defined by applying the divergence operator row-wise to the matrix; namely

$$\mathrm{div}_{\hat{\boldsymbol{x}}}A(\hat{\boldsymbol{x}}) = \begin{bmatrix} \nabla_{\hat{\boldsymbol{x}}} \cdot \boldsymbol{a}_1(\hat{\boldsymbol{x}}) \\ \vdots \\ \nabla_{\hat{\boldsymbol{x}}} \cdot \boldsymbol{a}_n(\hat{\boldsymbol{x}}) \end{bmatrix},$$

which results in a column vector. This follows from the integration by part as follows:

$$\begin{aligned}
-\int_{\Omega} \mathrm{div}_{\hat{\boldsymbol{x}}}A \cdot \boldsymbol{\phi} &= \int_{\Omega} A : \nabla\boldsymbol{\phi} \\
&= \int_{\Omega} A : \left[\frac{\partial \phi_i}{\partial X_j}\right]_{ij} \\
&= \sum_{i,j} \int_{\Omega} a_{ij} \frac{\partial \phi_i}{\partial X_j} \\
&= -\sum_{i,j} \int_{\Omega} \frac{\partial a_{ij}}{\partial X_j} \phi_i \\
-\int_{\Omega} \mathrm{div}_{\hat{\boldsymbol{x}}}A \cdot \boldsymbol{\phi} &= -\int_{\Omega} \sum_{i} \left(\sum_{j} \frac{\partial a_{ij}}{\partial X_j}\right) \phi_i
\end{aligned}$$

Chain Rule

For vector and matrix functions in Eulerian coordinates, the derivatives are define analogously to the derivatives above. Furthermore, for a scalar function $a(\boldsymbol{x})$, we have the following chain rule:

$$\nabla_{\boldsymbol{x}} a(\boldsymbol{x}) = \left[\frac{\partial a}{\partial \boldsymbol{x}_i}\right]_i = \left[\sum_{j=1}^{n} \frac{\partial \hat{\boldsymbol{x}}_j}{\partial \boldsymbol{x}_i} \frac{\partial a}{\partial \hat{\boldsymbol{x}}_j}\right]_i = \hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}}, t)\nabla_{\hat{\boldsymbol{x}}} a(\boldsymbol{x}(\hat{\boldsymbol{x}}, t)).$$

We also have a chain rule for vector functions $\boldsymbol{\alpha}(\boldsymbol{x})$:

$$\nabla_{\boldsymbol{x}} \boldsymbol{\alpha}(\boldsymbol{x}) = \left[\frac{\partial \alpha_i}{\partial \boldsymbol{x}_j}\right]_{ij} = \left[\sum_{j=1}^{n} \frac{\partial \alpha_i}{\partial \hat{\boldsymbol{x}}_k} \frac{\partial \hat{\boldsymbol{x}}_k}{\partial \boldsymbol{x}_j}\right]_{ij} = \nabla_{\hat{\boldsymbol{x}}} \boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t))\hat{\boldsymbol{F}}^{-1}(\hat{\boldsymbol{x}}, t).$$

14.1.1 Kinematics

Flow map

Consider a subdomain $\Omega(0)$ of the domain of a mechanical system and suppose we have the *flow map* $\boldsymbol{x}(\hat{\boldsymbol{x}}, t) : \Omega(0) \times [0, T] \mapsto \Omega(t)$. We call $\Omega(t)$ a control volume, which is parametrized by the time variable t, and we refer to $\Omega(0)$ as the reference volume. For $(\hat{\boldsymbol{x}}, t) \in \Omega(0) \times [0, T]$, define

- displacement:
$$\hat{\boldsymbol{u}}(\hat{\boldsymbol{x}}, t) = \boldsymbol{x}(\hat{\boldsymbol{x}}, t) - \hat{\boldsymbol{x}},$$
- velocity:
$$\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}, t) = \frac{\partial \hat{\boldsymbol{x}}}{\partial t},$$
- deformation tensor:
$$\hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) = \frac{\partial \boldsymbol{x}}{\partial \hat{\boldsymbol{x}}}(\hat{\boldsymbol{x}}, t), \qquad \left(\hat{\boldsymbol{F}}_{ij} = \frac{\partial \boldsymbol{x}_i}{\partial \hat{\boldsymbol{x}}_j}, \quad \hat{J}(\hat{\boldsymbol{x}}, t) = det(\hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t))\right).$$

The inverse of the flow map, $\hat{\boldsymbol{x}}(\boldsymbol{x}, t)$, can be defined only if $\hat{J} = det(\hat{\boldsymbol{F}}) \neq 0$.

The variables above are defined in *Lagrangian coordinates*, $(\hat{\boldsymbol{x}}, t)$. We may also define variables in *Eulerian coordinates*, $(\boldsymbol{x}, t)$:

- velocity:
$$\boldsymbol{v}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) = \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}, t),$$
- deformation tensor:
$$\boldsymbol{F}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) = \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) \quad \text{and} \quad J(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) = \hat{J}(\hat{\boldsymbol{x}}, t).$$

Material Derivatives

Consider any smooth function $\boldsymbol{\alpha}(\boldsymbol{x}, t)$, given in Eulerian coordinates. We write α in terms of the Lagrangian coordinates, $\boldsymbol{x} = \boldsymbol{x}(\hat{\boldsymbol{x}}, t)$, and take the time derivative to define the *material derivative*,

$$\begin{aligned} \frac{D\boldsymbol{\alpha}}{Dt}(\boldsymbol{x}, t) &= \frac{\partial[\boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)]}{\partial t} \\ &= \boldsymbol{\alpha}_t(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) + \nabla_{\boldsymbol{x}}\boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\boldsymbol{x}_t(\hat{\boldsymbol{x}}, t) \\ &= \boldsymbol{\alpha}_t(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) + \nabla_{\boldsymbol{x}}\boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\boldsymbol{v}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) \\ &\equiv \boldsymbol{\alpha}_t(\boldsymbol{x}, t) + (\boldsymbol{v} \cdot \nabla_{\boldsymbol{x}})\boldsymbol{\alpha}(\boldsymbol{x}, t). \end{aligned} \tag{14.1}$$

Given the function in Lagrangian coordinates, $\hat{\boldsymbol{\alpha}}(\hat{\boldsymbol{x}}, t) = \boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)$, it follows by definition that

$$\frac{\partial \hat{\boldsymbol{\alpha}}}{\partial t}(\hat{\boldsymbol{x}}, t) = \frac{D\boldsymbol{\alpha}}{Dt}(\boldsymbol{x}, t)\Big|_{\boldsymbol{x}=\boldsymbol{x}(\hat{\boldsymbol{x}}, t)}.$$

Lemma 102.
$$\frac{\partial \hat{J}}{\partial t} = (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v})\hat{J}$$

proof

To compute the time derivative of $\hat{J}$, we use cofactor expansion and write

$$\hat{J}(\hat{\boldsymbol{x}}, t) = \det \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) = \sum_{k=1}^{n} \hat{\boldsymbol{F}}_{i,k}(\hat{\boldsymbol{x}}, t) \det \hat{\boldsymbol{F}}^c_{i,k}(\hat{\boldsymbol{x}}, t),$$

where $\hat{\boldsymbol{F}}^c_{i,k}$ is the cofactor matrix of $\hat{\boldsymbol{F}}_{i,k}$, attained by deleting the i^{th} row and k^{th} column. (Note that the choice of expanding along row i is permissible, since cofactor expansion is row independent.) Consequently, we have

$$
\begin{aligned}
\frac{\partial}{\partial \hat{\boldsymbol{F}}_{i,j}} \hat{J} &= \frac{\partial}{\partial \hat{\boldsymbol{F}}_{i,j}} \sum_{k=1}^{n} \hat{\boldsymbol{F}}_{i,k} (-1)^{i+k} \det \hat{\boldsymbol{F}}_{i,k}^{c} \\
&= \sum_{k=1}^{n} \left[\frac{\partial \hat{\boldsymbol{F}}_{i,k}}{\partial \hat{\boldsymbol{F}}_{i,j}} (-1)^{i+k} \det \hat{\boldsymbol{F}}_{i,k}^{c} + \hat{\boldsymbol{F}}_{i,k} \frac{\partial}{\partial \hat{\boldsymbol{F}}_{i,j}} (-1)^{i+k} \det \hat{\boldsymbol{F}}_{i,k}^{c} \right] \\
&= (-1)^{i+j} \det \hat{\boldsymbol{F}}_{i,j}^{c},
\end{aligned}
$$

where we note $\partial(\det \hat{\boldsymbol{F}}_{i,k}^{c})/\partial \hat{\boldsymbol{F}}_{i,j} = 0$ because $\hat{\boldsymbol{F}}_{i,k}^{c}$ is $\hat{\boldsymbol{F}}$ with the i^{th} row and k^{th} column removed. The above calculation shows that the derivative of the determinant with respect to the matrix yields the *cofactor* matrix of $\hat{\boldsymbol{F}}$,

$$
\frac{\partial}{\partial \hat{\boldsymbol{F}}} \hat{\boldsymbol{J}} = \frac{\partial}{\partial \hat{\boldsymbol{F}}} \det \hat{\boldsymbol{F}} = [(-1)^{i+j} \det \hat{\boldsymbol{F}}_{i,j}^{c}]_{i,j} = \hat{\boldsymbol{J}} \hat{\boldsymbol{F}}^{-1},
$$

where $\partial \hat{\boldsymbol{J}} / \partial \hat{\boldsymbol{F}}$ is a matrix of derivatives with its ij^{th} entry as $\partial \hat{J} / \partial \hat{\boldsymbol{F}}_{i,j}$.

The derivative of determinant can be calculated as follows:

$$
\begin{aligned}
\frac{\partial}{\partial t} \hat{J}(\hat{\boldsymbol{x}}, t) &= \frac{\partial}{\partial t} \det \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) \\
\text{(Chain Rule for Matrices)} \quad &= tr\left(\left(\frac{\partial}{\partial \hat{\boldsymbol{F}}} \det \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) \right)^{T} \frac{\partial}{\partial t} \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) \right) \\
&= tr\left(\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}}, t) \frac{\partial}{\partial t} \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) \right) \hat{J}(\hat{\boldsymbol{x}}, t) \\
&= tr\left(\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}}, t) \frac{\partial^2}{\partial \hat{\boldsymbol{x}} \partial t} \boldsymbol{x}(\hat{\boldsymbol{x}}, t) \right) \hat{J}(\hat{\boldsymbol{x}}, t) \\
\text{(Inverse Function Theorem)} \quad &= tr\left(\frac{\partial \hat{\boldsymbol{x}}^{T}}{\partial \boldsymbol{x}} (\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) \frac{\partial}{\partial \hat{\boldsymbol{x}}} \hat{\boldsymbol{v}}(\hat{\boldsymbol{x}}, t) \right) \hat{J}(\hat{\boldsymbol{x}}, t) \\
\text{(Chain Rule)} \quad &= tr\left(\frac{\partial}{\partial \boldsymbol{x}} \boldsymbol{v}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) \right) \hat{J}(\hat{\boldsymbol{x}}, t) \\
\text{(14.2)} \quad &= (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)) \hat{J}(\hat{\boldsymbol{x}}, t).
\end{aligned}
$$

Alternate derivation of $\partial \hat{J} / \partial t$:

To compute the time derivative of $\hat{J}$, we have

$$
\begin{aligned}
\frac{\partial}{\partial t} \hat{J}(\hat{\boldsymbol{x}}, t) = \frac{\partial}{\partial t} \det \hat{\boldsymbol{F}}(\hat{\boldsymbol{x}}, t) &= \frac{\partial}{\partial t} \det \begin{bmatrix} (\nabla_{\hat{\boldsymbol{x}}} x_1(\hat{\boldsymbol{x}}, t))^{T} \\ \vdots \\ (\nabla_{\hat{\boldsymbol{x}}} x_n(\hat{\boldsymbol{x}}, t))^{T} \end{bmatrix} \\
&= \frac{\partial}{\partial t} \det \left[\nabla_{\hat{\boldsymbol{x}}} x_1(\hat{\boldsymbol{x}}, t) \cdots \nabla_{\hat{\boldsymbol{x}}} x_n(\hat{\boldsymbol{x}}, t) \right] \\
&= \sum_{i=1}^{n} \det \left[\nabla_{\hat{\boldsymbol{x}}} x_1(\hat{\boldsymbol{x}}, t) \cdots \nabla_{\hat{\boldsymbol{x}}} \hat{v}_i(\hat{\boldsymbol{x}}, t) \cdots \nabla_{\hat{\boldsymbol{x}}} x_n(\hat{\boldsymbol{x}}, t) \right].
\end{aligned}
$$

We evaluate the determinants by cofactor expansion along the i^{th} column, where the cofactor F_{ij}^{c} is attained by deleting the i^{th} column and j^{th} row. Thus,

$$\frac{\partial \hat{J}}{\partial t} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial v_i}{\partial X_j}(-1)^{i+j} \det F_{ij}^c$$

(Chain Rule)

$$= \sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} \frac{\partial v_i}{\partial x_k}\frac{\partial x_k}{\partial X_j}\Big)(-1)^{i+j} \det F_{ij}^c$$

$$= \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial v_i}{\partial x_k}\sum_{j=1}^{n}\frac{\partial x_k}{\partial X_j}(-1)^{i+j} \det F_{ij}^c$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial v_i}{\partial x_i}\frac{\partial x_i}{\partial X_j}(-1)^{i+j} \det F_{ij}^c \tag{14.3}$$

$$= (\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v})\hat{J}, \tag{14.4}$$

where we used $\sum_{j=1}^{n} \frac{\partial x_k}{\partial X_j} \det F_{ij}^c = 0$ for $i \neq k$ to get (14.3).

A Transport Identity

We present a lemma, sometimes referred to as Reynolds' Transport Theorem, that is essential to proving conservation laws.

Lemma 103. *Suppose $\Omega(t)$ is a control volume evolves according to the flow map $\boldsymbol{x}(\hat{\boldsymbol{x}}, t)$ given above and that the deformation tensor $\boldsymbol{F}$ is invertible for all $(\hat{\boldsymbol{x}}, t)$. Then for any differentiable vector function $\boldsymbol{\alpha}(\boldsymbol{x}, t)$ in $\Omega(t)$, we have*

$$\frac{d}{dt}\int_{\Omega(t)} \boldsymbol{\alpha}(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\Omega(t)} \boldsymbol{\alpha}_t(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{v}^T(\boldsymbol{x}, t))\, d\boldsymbol{x}.$$

To prove this lemma, we begin with a change of variables to Lagrangian coordinates, so that time derivative may be passed into the integrand:

$$\begin{aligned} \frac{d}{dt}\int_{\Omega(t)} \boldsymbol{\alpha}(\boldsymbol{x}, t) d\boldsymbol{x} &= \frac{d}{dt}\int_{\Omega(0)} \boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\hat{J}(\hat{\boldsymbol{x}}, t) d\hat{\boldsymbol{x}} \\ &= \int_{\Omega(0)} \frac{d}{dt}(\boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t))\hat{J}(\hat{\boldsymbol{x}}, t) + \boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\frac{d}{dt}\hat{J}(\hat{\boldsymbol{x}}, t) d\hat{\boldsymbol{x}}. \end{aligned} \tag{14.5}$$

From the last section, we see that

$$\frac{d}{dt}\boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t) = \frac{D}{Dt}\boldsymbol{\alpha}(\boldsymbol{x}, t)\Big|_{\boldsymbol{x}=\boldsymbol{x}(\hat{\boldsymbol{x}},t)} = \boldsymbol{\alpha}_t(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}}\boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t). \tag{14.6}$$

Combining (14.4)–(14.6) followed by a change of variables back to Eulerian coordinates yields

$$\begin{aligned} \frac{d}{dt}\int_{\Omega(t)} \boldsymbol{\alpha}(\boldsymbol{x}, t) d\boldsymbol{x} &= \int_{\Omega(0)} \frac{D}{Dt}\boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\hat{J}(\hat{\boldsymbol{x}}, t) + \boldsymbol{\alpha}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\hat{J}(\hat{\boldsymbol{x}}, t))\, d\hat{\boldsymbol{x}} \\ &= \int_{\Omega(t)} \boldsymbol{\alpha}_t(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}}\boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t) + \boldsymbol{\alpha}(\boldsymbol{x}, t)(\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}, t))\, d\boldsymbol{x}, \\ &= \int_{\Omega(t)} \boldsymbol{\alpha}_t(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{v}^T(\boldsymbol{x}, t))\, d\boldsymbol{x}, \end{aligned}$$

where the last step follows from the product rule.

Note that in case that $\boldsymbol{\alpha}$ is a scalar, we write the divergence term as $\mathrm{div}_{\boldsymbol{x}}(\boldsymbol{\alpha}(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t))$.

14.1.2 Two Fundamental Laws of Mechanics

Equations for most mechanical systems can derived from two fundamental physical laws: the conservation of mass and the balance of linear momentum. In the following sections, we formulate these two laws mathematically and introduce Cauchy's Stress Theorem.

Conservation of Mass

A fundamental law for mechanical systems is the *conservation of mass*, which states that the total mass of a physical entity does not change in time. Let $\rho(\boldsymbol{x}, t)$ represent the density of some physical entity in Eulerian coordinates and suppose this density evolves in time via the flow map, $\boldsymbol{x}(\hat{\boldsymbol{x}}, t)$, described above.

Consider this density function restricted to some control volume, $\Omega(t)$ at time t, that also evolves with the flow map. To be precise, this means that all points $\hat{\boldsymbol{x}} \in \Omega(0)$ at time $t = 0$ remain inside the control volume: $\boldsymbol{x}(\hat{\boldsymbol{x}}, t) \in \Omega(t)$. Let $m(t)$ represent the total mass in $\Omega(t)$. Simply stated, the conservation of mass requires for each control volume $\Omega(t)$

$$\frac{d}{dt}m(t) = \frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x}, t)\, d\boldsymbol{x} = 0. \tag{14.7}$$

To derive the differential form of the conservation of mass, we invoke lemma 103 with $\boldsymbol{\alpha} = \rho$ to show

$$\begin{aligned} 0 &= \frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &= \int_{\Omega(t)} \rho_t(\boldsymbol{x}, t) + \operatorname{div}(\rho(\boldsymbol{x}, t)\boldsymbol{v}^T(\boldsymbol{x}, t))\, d\boldsymbol{x} \\ &= \int_{\Omega(t)} \rho_t(\boldsymbol{x}, t) + \nabla \cdot (\rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t))\, d\boldsymbol{x}, \end{aligned}$$

where the last equality follows from the density ρ being a scalar function. Since mass is conserved over all control volume subsets of our mechanical system's domain, we recover the following partial differential equation,

$$\rho_t(\boldsymbol{x}, t) + \nabla \cdot (\rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)) = 0. \tag{14.8}$$

If a density function is given in *Lagrangian coordinates* (in terms of $\hat{\boldsymbol{x}}$), then we have

$$\hat{\rho}(\hat{\boldsymbol{x}}, t) = \rho(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t).$$

The conservation of mass can translated to this case using the following reasoning:

$$\begin{aligned} &\text{(definition of flow map)} && \int_{\Omega(0)} \hat{\rho}(\hat{\boldsymbol{x}}, 0)\, d\hat{\boldsymbol{x}} = \int_{\Omega(0)} \rho(\boldsymbol{x}(\hat{\boldsymbol{x}}, 0), 0)\, d\hat{\boldsymbol{x}} \\ &\text{(conservation of mass)} && \qquad = \int_{\Omega(t)} \rho(\boldsymbol{x}, t)\, d\boldsymbol{x} \\ &\text{(change of variables)} && \qquad = \int_{\Omega(0)} \rho(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t)\hat{J}(\hat{\boldsymbol{x}}, t)\, d\hat{\boldsymbol{x}} \\ & && \qquad = \int_{\Omega(0)} \hat{\rho}(\hat{\boldsymbol{x}}, t)\hat{J}(\hat{\boldsymbol{x}}, t)\, d\hat{\boldsymbol{x}}, \end{aligned}$$

which shows that the identity

$$\hat{\rho}(\hat{\boldsymbol{x}}, t) = \frac{\hat{\rho}(\hat{\boldsymbol{x}}, 0)}{\hat{J}(\hat{\boldsymbol{x}}, t)} \tag{14.9}$$

is the statement for mass conservation in Lagrangian coordinates. Since $\hat{J}(\hat{\boldsymbol{x}}, t)$ measures the infinitesimal volume at $\boldsymbol{x}(\hat{\boldsymbol{x}}, t) \in \Omega(t)$, equation (14.10) states that the evolution of the density is solely determined by the initial density configuration and the change in volume induced over time by the flow map.

or we can write

$$\rho(x, t)dx = \hat{\rho}(\hat{x}, 0)d\hat{x} \tag{14.10}$$

Balance of Linear Momentum and Cauchy's Stress Theorem

In a mechanical system, a body inside a control volume can experience external forces as *stress* on the surface of the volume. Often, mechanical bodies experience more than one type of stress at the same time; this is called combined stress. Combined stresses cannot be described by a single vector, even if the material is stressed in the same way throughout the volume of the body. The stress across the surface of a control volume may depend on the orientation of the surface in a non-trivial way. We represent the stress vector by $\boldsymbol{t}(\boldsymbol{x}, t, \partial\Omega)$, the dependence of the stress vector on $\partial\Omega$ allows the stress vector to depend upon the orientation of the volume's surface.

Assume that the *balance of linear momentum*—Newton's second law of motion—holds over all control volumes, then we may write

$$\frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\Omega(t)} \boldsymbol{f}(\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial\Omega(t)} \boldsymbol{t}(\boldsymbol{\xi}, t, \partial\Omega)\, d\boldsymbol{\xi}, \tag{14.11}$$

where $\rho\boldsymbol{v}$ is the body momentum and $\boldsymbol{f}$ represents any body force acting within the control volume.

We are interested in understanding the properties of the stress vector. Namely, Cauchy's Stress Theorem states that the stress vector acting upon a body is a linear function of the unit normal vector to the body's surface — though it may have nonlinear dependence on the coordinates of the surface.

Theorem 102 (Cauchy's Stress Theorem). *Assume that the balance of linear momentum (14.11) holds over all control volumes. Then, there is a symmetric second-order stress tensor $\boldsymbol{\sigma}(\boldsymbol{x}, t)$ such that the stress vector for any control volume is a linear function of the normal vector,*

$$\boldsymbol{T}(\boldsymbol{x}, t, \partial\Omega) = \boldsymbol{\sigma}(\boldsymbol{x}, t)\boldsymbol{n} \quad \text{or} \quad T_i(\boldsymbol{x}, t, \partial\Omega) = \sum_{j=1}^{3} \sigma_{i,j}(\boldsymbol{x}, t)n_j. \tag{14.12}$$

Given stress in Eulerian coordinates: $\boldsymbol{\sigma}(\boldsymbol{x}, t)$, we can define the corresponding Lagrangian stress

$$\hat{\boldsymbol{\sigma}}(\hat{\boldsymbol{x}}, t) := \boldsymbol{\sigma}(\boldsymbol{x}(\hat{\boldsymbol{x}}, t), t).$$

Stress is a physical quantity that expresses the internal forces that neighboring particles of a continuous material exert on each other.

Applying (14.12) to (14.11) gives rise to

$$\frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\Omega(t)} \boldsymbol{f}(\boldsymbol{x}, t)\, d\boldsymbol{x} + \int_{\partial\Omega(t)} \boldsymbol{\sigma}(\boldsymbol{\xi}, t)\boldsymbol{n}\, d\boldsymbol{\xi}. \tag{14.13}$$

Apply the divergence theorem to the stress term to obtain

$$\frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\, d\boldsymbol{x} = \int_{\Omega(t)} \boldsymbol{f}(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}}\boldsymbol{\sigma}(\boldsymbol{x}, t)\, d\boldsymbol{x}. \tag{14.14}$$

As this holds for arbitrary control volumes, we apply lemma 103 to the term on the left to show

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t}(\boldsymbol{x},t) + \mathrm{div}_{\boldsymbol{x}}(\rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\boldsymbol{v}^T(\boldsymbol{x},t)) = \boldsymbol{f}(\boldsymbol{x},t) + \mathrm{div}_{\boldsymbol{x}}\boldsymbol{\sigma}(\boldsymbol{x},t), \quad \text{in } \Omega(t). \tag{14.15}$$

Based on Equation (14.8), it can be further simplified as

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}\right) = \boldsymbol{f}(\boldsymbol{x},t) + \mathrm{div}_{\boldsymbol{x}}\boldsymbol{\sigma}(\boldsymbol{x},t), \quad \text{in } \Omega(t). \tag{14.16}$$

Note that

$$A : (BC) = \mathrm{trace}(A^TBC) = \mathrm{trace}(C^TB^TA) = \mathrm{trace}((AC^T)^TB) = (AC^T) : B.$$

We will use this identity in the proof of the next lemma.

Lemma 104. *For any symmetric tensor σ,*

$$\mathrm{div}_x\sigma = \hat{J}^{-1}\mathrm{div}_{\hat{x}}(\hat{J}\hat{\sigma}\hat{F}^{-T}). \tag{14.17}$$

Proof:

$\forall \phi \in C_0^\infty$, by divergence theorem,

$$\begin{aligned}
&\int_{\Omega(t)} \mathrm{div}_{\boldsymbol{x}}\sigma\cdot\phi d\boldsymbol{x}\\
=&-\int_{\Omega(t)} \sigma : \nabla\phi d\boldsymbol{x}\\
=&-\int_{\Omega(t)} \sigma : (\nabla_{\hat{\boldsymbol{x}}}\phi F^{-1})d\boldsymbol{x}\\
=&-\int_{\Omega(0)} \hat{\sigma} : (\nabla_{\hat{\boldsymbol{x}}}\hat{\phi}\hat{F}^{-1}\hat{J})d\hat{\boldsymbol{x}}\\
=&-\int_{\Omega(0)} (J\sigma\hat{F}^{-T}) : \nabla_{\hat{\boldsymbol{x}}}\hat{\phi}d\hat{\boldsymbol{x}}\\
=&\int_{\Omega(0)} \mathrm{div}_{\hat{\boldsymbol{x}}}(J\hat{\sigma}\hat{F}^{-T})\cdot\hat{\phi}d\hat{\boldsymbol{x}}\\
=&\int_{\Omega(t)} \hat{J}^{-1}\mathrm{div}_{\hat{\boldsymbol{x}}}(J\hat{\sigma}\hat{F}^{-T})\cdot\phi d\boldsymbol{x}.
\end{aligned} \tag{14.18}$$

Since ϕ is arbitrary, we know that the equation (14.17) holds.

Lagrangian Formulation

To derive a Lagrangian counterpart of equation (14.23), we use the conservation of mass (14.10),

$$\hat{\rho}(\hat{\boldsymbol{x}},t) = \hat{\rho}(\hat{\boldsymbol{x}})/\hat{J}(\hat{\boldsymbol{x}},t),$$

Thus

$$\int_{\Omega(t)} \rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\,d\hat{x} = \int_{\Omega(0)} \hat{\rho}(\hat{x},t)\hat{\boldsymbol{v}}(\hat{x},t)\hat{J}(\hat{x},t)\,d\hat{x} = \int_{\Omega(0)} \hat{\rho}(\hat{\boldsymbol{x}})\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}},t)\,d\hat{\boldsymbol{x}},$$

for the term on the left in equation (14.13),

$$\frac{d}{dt}\int_{\Omega(t)} \rho(\boldsymbol{x},t)\boldsymbol{v}(\boldsymbol{x},t)\,d\boldsymbol{x} = \int_{\Omega(0)} \hat{\rho}(\hat{\boldsymbol{x}})\frac{d}{dt}\hat{\boldsymbol{v}}(\hat{\boldsymbol{x}},t)\,d\hat{\boldsymbol{x}}, \tag{14.19}$$

where $\hat{\rho}(\hat{\boldsymbol{x}}) = \hat{\rho}(\hat{\boldsymbol{x}},0)$. For the stress term in (14.13), we perform a change of variables and note that

$$\boldsymbol{n}(\boldsymbol{x}(\hat{\boldsymbol{x}},t))d\boldsymbol{x} = \hat{J}(\hat{\boldsymbol{x}},t)\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}},t)\hat{\boldsymbol{n}}(\hat{\boldsymbol{x}})d\hat{\boldsymbol{x}}.$$

Hence,

$$\begin{aligned}
\int_{\partial\Omega(t)} \boldsymbol{\sigma}(\boldsymbol{x},t)\boldsymbol{n}(\boldsymbol{x},t)\,d\boldsymbol{x} &= \int_{\partial\Omega(0)} \boldsymbol{\sigma}(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t)\Big(\hat{J}(\hat{\boldsymbol{x}},t)\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}},t)\hat{\boldsymbol{n}}(\hat{\boldsymbol{x}})\Big)\,d\hat{\boldsymbol{x}}\\
&= \int_{\partial\Omega(0)} \Big(\hat{J}(\hat{\boldsymbol{x}},t)\boldsymbol{\sigma}(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t)\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}},t)\Big)\hat{\boldsymbol{n}}(\hat{\boldsymbol{x}})\,d\hat{\boldsymbol{x}}\\
&= \int_{\Omega(0)} \mathrm{div}_{\hat{\boldsymbol{x}}}\Big(\hat{J}(\hat{\boldsymbol{x}},t)\boldsymbol{\sigma}(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t)\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}},t)\Big)\,d\hat{\boldsymbol{x}}
\end{aligned} \tag{14.20}$$

Let $\hat{\boldsymbol{\sigma}}(\hat{\boldsymbol{x}},t) = \boldsymbol{\sigma}(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t)$ and $\hat{\boldsymbol{f}}(\hat{\boldsymbol{x}},t) = \boldsymbol{f}(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t)$ and we use (14.13), (14.19)–(14.20) to get

$$\int_{\Omega(0)} \hat{\rho}(\hat{\boldsymbol{x}})\frac{\partial\hat{\boldsymbol{v}}}{\partial t}(\hat{\boldsymbol{x}},t)\,d\hat{\boldsymbol{x}} = \int_{\Omega(0)} \hat{J}(\hat{\boldsymbol{x}},t)\hat{\boldsymbol{f}}(\hat{\boldsymbol{x}},t) + \mathrm{div}_{\hat{\boldsymbol{x}}}\Big(\hat{J}(\hat{\boldsymbol{x}},t)\boldsymbol{\sigma}(\boldsymbol{x}(\hat{\boldsymbol{x}},t),t)\hat{\boldsymbol{F}}^{-T}(\hat{\boldsymbol{x}},t)\Big)\,d\hat{\boldsymbol{x}}.$$

Since this holds over arbitrary control volumes, it follows that

$$\hat{\rho}\frac{\partial\hat{\boldsymbol{v}}}{\partial t} = \hat{J}\hat{\boldsymbol{f}} + \nabla_{\hat{\boldsymbol{x}}}\cdot(\hat{J}\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{F}}^{-T}). \tag{14.21}$$

Constitutive Equations

Constitutive equations establish mathematical relationship among the statical, kinematical and thermal variables for material subject to mechanical or thermal forces[c0], such as the relation between strain and stress in elastic solid and the relation between rate of strain and stress in fluid. In addition, the following are also constitutive equations:

- Darcy's law: the proportionality between velocity and pressure gradient
- Ohm's law: the proportionality between potential difference and electrical current
- Fourier's law: the proportionality between heat flux and temperature gradient.

Note that real materials may have extremely complicated response under loadings and constitutive equations cannot cover all the observed data. They can only fit the data reasonably well over a range of physical parameters.

We introduce constitutive equations via the concept of dynamical process $(\boldsymbol{x},\sigma)$, which is a pair of trajectory $\boldsymbol{x}$ and Cauchy stress tensor σ. Constitutive equations are about how σ is related to $\boldsymbol{x}$.

A dynamic process $(\boldsymbol{x},\sigma)$ is isochoric if

$$\mathrm{vol}(\boldsymbol{x}(\hat{V},t)) = \mathrm{vol}(\hat{V}), \quad \forall \hat{V}\subset\hat{\Omega}$$

holds for any $t > 0$.

A material body is a body Ω with a mass distribution and a family C of dynamical processes. C is called the constitutive class of the material body. A material body is incompressible if each $(\boldsymbol{x},\sigma)\in C$ is isochoric.

[c0] Schaum's outline of continuum mechanics

Incompressibility (Volume preserving)

For incompressible materials, the volume of the material does not change over time:

$$\frac{d}{dt}\int_{\Omega(t)} 1\, d\boldsymbol{x} = \int_{\Omega(0)} \frac{d}{dt}\hat{J}(\hat{\boldsymbol{x}}, t)\, d\hat{\boldsymbol{x}} = 0.$$

Therefore, we have $\hat{J} \equiv 1$ in $\Omega(0)$, which is equivalent to $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{v} = 0$ in $\Omega(t)$ by equation (14.4). Also by equation (14.8), $D\rho/Dt \equiv 0$ for incompressible materials.

Constitutive equations of incompressible viscous fluids

A **Newtonion fluid** is a material body consistent with the following assumptions:

- There exists linear function C such that the constitutive class is the set of all isochoric dynamical processes $(\boldsymbol{x}, \sigma)$ which satisfies
$$\sigma = -pI + C(\nabla \boldsymbol{v}).$$
- The density ρ_0 is constant.

Theorem 103. *The response of a Newtonian fluid is independent of the observer if and only if $C(\nabla \boldsymbol{v}) = 2\mu D$, where*
$$D = \frac{1}{2}(\nabla \boldsymbol{v} + (\nabla \boldsymbol{v})^T).$$
μ is called the viscosity of the fluid.

Based on conservation of momentum, we have

$$\frac{\partial(\rho \boldsymbol{v})}{\partial t}(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}}(\rho(\boldsymbol{x}, t)\boldsymbol{v}(\boldsymbol{x}, t)\boldsymbol{v}^T(\boldsymbol{x}, t)) = \boldsymbol{f}(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}}\boldsymbol{\sigma}(\boldsymbol{x}, t), \quad \text{in } \Omega(t). \tag{14.22}$$

It can be further simplified as

$$\rho\frac{\partial \boldsymbol{v}}{\partial t}(\boldsymbol{x}, t) + \rho(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{f}(\boldsymbol{x}, t) + \mathrm{div}_{\boldsymbol{x}}\boldsymbol{\sigma}(\boldsymbol{x}, t), \quad \text{in } \Omega(t). \tag{14.23}$$

For Newtonian fluid, $\sigma = -pI + 2\mu D$. By some calculation, we know that

$$\mathrm{div}\sigma = -\nabla p + \mu\Delta \boldsymbol{v}.$$

Together with the continuity equation, we have the Navier-Stokes equations as follows

$$\begin{cases} \rho\dfrac{D\boldsymbol{v}}{Dt} - \mu\Delta \boldsymbol{v} + \nabla p = f \\ \mathrm{div}\boldsymbol{v} = 0 \end{cases} \tag{14.24}$$

Note that the momentum equations can also be written in the following conservative form

$$\frac{\partial}{\partial t}(\rho \boldsymbol{v}) - \mathrm{div}(2\mu\epsilon(\boldsymbol{v}) - \rho \boldsymbol{v}\boldsymbol{v}^T - p\mathbb{I}) = f.$$

Weak form of Navier-Stokes equations

Define

$$\mathbb{V} = \left\{ \boldsymbol{v} \in L^2(\Omega) | \nabla \boldsymbol{v} \in L^2(\Omega), v|_{\partial\Omega} = 0 \right\} = (H_0^1(\Omega))^n$$

and

$$\mathbb{Q} = L_0^2(\Omega) = \{ q \in L^2(\Omega), \int q = 0 \}.$$

Take any $\phi \in \mathbb{V}$. Note that

$$-\int_\Omega \Delta \boldsymbol{v} \cdot \phi = -\int_\Omega \mathrm{div}(\epsilon(\boldsymbol{v})) \cdot \phi = \int_\Omega \epsilon(\boldsymbol{v}) : \nabla \phi = \int_\Omega \epsilon(\boldsymbol{v}) : \epsilon(\phi).$$

The weak form of Navier-Stokes equations is as follows

$$\begin{cases} \left(\rho \dfrac{D\boldsymbol{v}}{Dt}, \phi\right) + 2\mu(\epsilon(\boldsymbol{v}), \epsilon(\phi)) - (p, \mathrm{div}\phi) = (f, \phi), & \forall \phi \in \mathbb{V} \\ (\mathrm{div}\boldsymbol{v}, q) = 0, & \forall q \in \mathbb{Q} \end{cases} \tag{14.25}$$

Dimensionless form of Navier-Stokes equations

We non-dimensionalize all the quantities with given characteristic length L, speed V, pressure P as follows:

$$\bar{t} = t\frac{V}{L}, \ \bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L}, \ \bar{\boldsymbol{v}} = \frac{\boldsymbol{v}}{V}, \ \bar{p} = \frac{p - P}{\rho V^2}, \ \bar{\mu} = \frac{\mu}{\rho V L}$$

Further using the same symbols, without the bar, for the non-dimensional quantities. The non-dimensionalized system with the choice of material relations

$$\begin{cases} \dfrac{D\bar{\boldsymbol{v}}}{D\bar{t}} - \dfrac{1}{Re}\Delta \bar{\boldsymbol{v}} + \bar{\nabla}\bar{p} = f \\ \bar{\nabla} \cdot \bar{\boldsymbol{v}} = 0 \end{cases} \quad \text{in } \Omega(t). \tag{14.26}$$

The Reynold's number Re is defined as

$$Re = \frac{1}{\bar{\mu}} = \frac{\rho V L}{\mu}.$$

Note that this procedure is only for the case where ρ is constant in Eulerian coordinate.

In the case where $Re << 1$, the Stokes equations can be obtained

$$\begin{cases} -\Delta \bar{\boldsymbol{v}} + \bar{\nabla}\tilde{p} = f \\ \bar{\nabla} \cdot \bar{\boldsymbol{v}} = 0 \end{cases} \quad \text{in } \Omega(t), \tag{14.27}$$

where $\tilde{p} = Re\bar{p}$.

A very similar type of equations is about the Darcy's Law for porous media:

$$\begin{cases} -\boldsymbol{v} + \dfrac{k}{\mu}\nabla p = f \\ \nabla \cdot \boldsymbol{v} = 0 \end{cases} \quad \text{in } \Omega(t). \tag{14.28}$$

Semi-Lagrangian method

Now we introduce notation $\boldsymbol{x}(\hat{\boldsymbol{x}}, t, s) = \boldsymbol{x}(\boldsymbol{x}^{-1}(\hat{\boldsymbol{x}}, t), s)$. By discretizing the time derivative as follows

$$
\begin{aligned}
\frac{D\boldsymbol{v}}{Dt}(\boldsymbol{x}, t^{n+1}) &= \frac{\partial}{\partial t}(\boldsymbol{v}(\hat{\boldsymbol{x}}(\boldsymbol{x}, t^{n+1}, t), t)) \\
&\approx \frac{\boldsymbol{v}(\hat{\boldsymbol{x}}(\boldsymbol{x}, t^{n+1}, t^{n+1}), t^{n+1}) - \boldsymbol{v}(\hat{\boldsymbol{x}}(\boldsymbol{x}, t^{n+1}, t^n), t^n)}{\Delta t} \\
&= \frac{\boldsymbol{v}(\boldsymbol{x}, t^{n+1}) - \boldsymbol{v}(\boldsymbol{x}^*, t^n)}{\Delta t},
\end{aligned}
\tag{14.29}
$$

we can obtain the Semi-Lagrangian method

$$
\begin{cases}
\dfrac{\boldsymbol{v}^{n+1} - \boldsymbol{v}_*^n}{\Delta t} - \mu\Delta\boldsymbol{v}^{n+1} + \nabla p^{n+1} = f^{n+1} \\
\mathrm{div}\boldsymbol{v}^{n+1} = 0
\end{cases}
\tag{14.30}
$$

Here $\boldsymbol{x}^* = \hat{\boldsymbol{x}}(\boldsymbol{x}, t^{n+1}, t^n)$ is the characteristic foot of $\boldsymbol{x}$.
We need to additionally solve the following equations

$$
\begin{cases}
\dfrac{\partial\hat{\boldsymbol{x}}}{\partial t} = \boldsymbol{v}(\hat{\boldsymbol{x}}, t) \\
\hat{\boldsymbol{x}}(\boldsymbol{x}, t^{n+1}, t^{n+1}) = \boldsymbol{x}.
\end{cases}
\tag{14.31}
$$

Non-Newtonian fluids

The Johnson-Seglman model:

$$
\begin{cases}
\mathrm{Re}\frac{Du}{Dt} = \nabla\cdot[\tau + \mu_s\mathcal{D}(u)] - \nabla p \\
\nabla\cdot u = 0 \\
\tau + Wi[\frac{D\tau}{Dt} - \mathcal{R}(u)\tau - \tau\mathcal{R}(u)^T] = 2\mu_p\mathcal{D}(u),
\end{cases}
$$

where $\frac{Du}{Dt} = \frac{\partial u}{\partial t} + (u\cdot\nabla)u, \quad \mathcal{D}(u) = \frac{1}{2}(\nabla u + \nabla u^T), \quad \mathcal{R}(u) = (\frac{a+1}{2}\nabla u + \frac{a-1}{2}\nabla u^T)$

If $a = 1$, it reduces to Oldryoid-B model.

The Wissenberg number Wi is very important parameter in this model. If $Wi = 0$, it becomes Newtonian fluid model. If $Wi = \infty$, the following transport equation holds

$$
\frac{D\tau_p}{Dt} + \nabla u\tau_p + \tau_p(\nabla u)^T.
$$

14.2 Stokes Equations

The Navier-Stokes equation reads:

$$
\begin{cases}
Re(\partial_t u + (u\cdot\nabla)p) - \Delta u + \nabla p = f \\
-\mathrm{div}u = 0.
\end{cases}
$$

When the Raynold constant tends to zero, we have

$$\begin{cases} -\Delta u + \nabla p = f \\ \quad -\mathrm{div} u = 0. \end{cases}$$

We introduce the Cauchy-stress

$$\sigma = e(u) - pI$$

with $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$. If a fluid satisfies the above condition, then it is called Newtwonian fluid. For an incompressible Newtwonian fluid,

$$\begin{cases} -\mathrm{div}\sigma = -\Delta u + \nabla p = f & \text{in } \Omega; \\ \qquad\qquad -\mathrm{div} u = 0 & \text{in } \Omega; \\ \qquad\qquad\qquad u = 0 & \text{on } \Gamma_D \\ \qquad e(u)n + pn = 0 & \text{on } \Gamma_T. \end{cases} \tag{14.32}$$

For simplicity, we assume that $\Gamma_D = \partial\Omega$ and define

$$V = (H_0^1(\Omega))^n, \quad Q = \{q \in L^2(\Omega) : \int_\Omega q = 0\}$$

with norms given by

$$|v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}, \quad |q|_{0,\Omega} = \|q\|_{0,\Omega}$$

respectively.

Multiply the first equation by v. By the boundary conditions in (14.32), we have

$$-(\mathrm{div}\sigma, v) = (\sigma, \nabla v) = (e(u), e(v)) - (pI, \nabla v) = (e(u), e(v)) - (p, \mathrm{div} v).$$

$$\begin{cases} a(u, v) + b(v, p) = (f, v) & \forall\ v \in V, \\ \qquad\quad b(u, q) = 0 & \forall\ q \in W, \end{cases} \tag{14.33}$$

where

$$a(u, v) = 2\mu \int_\Omega e(u) : e(v)\, dx, \quad b(v, q) = -\int_\Omega q \mathrm{div} v\, dx.$$

This stationary Stokes equation for an incompressible Newtwonian fluid with viscosity μ, enclosed in the domain $\Omega \in \mathbb{R}^2$, and acted upon by a volume load f can be characterised by the following minimization problem:

$$\inf_{v \in V, \mathrm{div} v = 0} \{\mu \int_\Omega |e(v)|^2 dx - \int_\Omega f \cdot v dx\}.$$

Here

$$V = (H_D^1(\Omega))^n, \quad (n = 2, 3).$$

The variable p can be viewed as the Lagrange multiplier associated with the incompressibility constraint.

Let $A = -\Delta$. Note that ∇ is the adjoint operator of $-\mathrm{div}$ due to the Stokes theorem:

$$-(\mathrm{div} v, q) = (v, \nabla q), \quad \forall v \in (H_0^1(\Omega))^d,\ q \in L^2(\Omega) \cap H_0^1(\Omega).$$

We have

$$B = -div : V \mapsto Q, \quad B^t = \nabla : Q \mapsto (H^{-1}(\Omega))^n.$$

Then the Stokes equation can be formally written as follows:

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{14.34}$$

The wellposedness of Stokes Equations are well studied. For example, inf-sup condition for B is valid since $B = -\mathrm{div}$ is onto, which is equivalent to $B^t = \nabla$ is one to one. We know that ∇ is indeed one-to-one in $(H_0^1)^d$.

14.2.1 Well-posedness and inf-sup condition

Formally, let us consider an LU factorization of the operator matrix for the Stokes equation:

$$\begin{pmatrix} A & B^t \\ B & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ -BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & B^t \\ 0 & -S \end{pmatrix}$$

where the Schur complement operator S is given by

$$S = BA^{-1}B^t : Q \mapsto Q. \tag{14.35}$$

Therefore the Stokes operator is an isomorphism if and only if the Schur compliment operator S is an isomorphism. First we show that S is a bounded operator.

$$\|Sq\|_0 = \| - \mathrm{div}(-\Delta)^{-1}\mathrm{grad}q\|_0 \lesssim \|(-\Delta)^{-1}\mathrm{grad}q\|_1$$

For any problem $-\Delta w = f$ with $f \in H^{-1}(\Omega)$, we have

$$|w|_1^2 = (f, w) \le \|f\|_{-1}\|w\|_1$$

meaning $|(-\Delta)^{-1}f|_1 \lesssim |f|_{-1}$. Thus,

$$\|Sq\|_0 \lesssim \|\mathrm{grad}q\|_{-1} = \sup_s \frac{(\mathrm{grad}q, s)}{\|s\|_1} = \sup_s \frac{(q, \mathrm{div}s)}{\|s\|_1} \le \|q\|_0.$$

It is also not difficult to see that S is onto. Now the question is whether S is invertible and has a bounded inverse, namely whether there is a constant $c > 0$ such that

$$|Sq|_0 \ge c|q|_0 \quad \forall\, q \in Q.$$

For $v \in V$ such that $-\Delta v = \nabla\phi$, we have

$$|v|_1^2 = (\nabla\phi, v) = (\phi, -\mathrm{div}v) \lesssim \|\phi\|_0|v|_1 \Rightarrow |v|_1 \lesssim \|\phi\|_0.$$

. Thus,

$$|Sq|_0 = \sup_{\phi\in W} \frac{(S\phi, q)}{|\phi|_0} = \sup_{\phi\in W} \frac{((-\Delta)^{-1}\mathrm{grad}\phi, \mathrm{grad}q)}{|\phi|_0} = \sup_{\phi\in W} \frac{(v, \mathrm{grad}q)}{|\phi|_0} \ge C \sup_{v\in V} \frac{b(v, q)}{|v|_1}$$

Therefore S is an isomorphism if there exists constant $\alpha > 0$ such that

$$\inf_{q\in W} \sup_{v\in V} \frac{b(v, q)}{|v|_1|q|_0} \ge \alpha > 0. \tag{14.36}$$

This is known as an inf-sup condition. This condition can be indeed proved rigorously. The details of the proof is omitted here.

Remark 31. 1. If $\Gamma_D = \partial\Omega$, $(e(u), e(v)) = (\nabla u, \nabla v)$. Thus,

$$\|e(u)\|_{0,\Omega} = |\nabla u|_{0,\Omega}.$$

Poincare inequality tells $\|u\|_1 \lesssim \|e(u)\|_0$ if $u \in H_0^1(\Omega)$.

2. If $\Gamma_D \ne \partial\Omega$ and $|\Gamma_D| \ne 0$, by Korn;s inequality,

$$\|u\|_1 \lesssim \|e(u)\|_0.$$

(3) In 3-dimension, if $e(v) = 0$, we have
$$v = a + b \times x.$$

For the special case $\Gamma_D = \partial\Omega$, we prove the well poseness of this problem.

Theorem 104. *Assume that the domain Ω is convex, the variational problem* (14.33) *is well-posed if $\Gamma_D = \partial\Omega$.*

Proof. By Brezzi's theorem, we can prove the theorem by proving that $a(\cdot,\cdot)$ is coercive and $b(\cdot,\cdot)$ satisfies the inf-sup condition. It is obvious that $a(u,u) \geq C\|u\|_{\mathrm{div}}$ for $u \in Ker(B)$.

For the inf-sup condition concerning $b(\cdot,\cdot)$, for any $q \in Q$, consider the auxiliary problem

$$\begin{aligned} -\Delta\phi &= q \quad \text{In } \Omega \\ \phi &= 0 \quad \text{on } \Gamma_D \end{aligned}$$

Take $u = \nabla\phi \in Q$, it is then easy to see that

$$(\mathrm{div} u, q) = \|q\|^2.$$

Since $\|u\|_V^2 = \|\nabla\phi\|_1^2 \leq C\|q\|_0^2$,

$$\sup_{v\in V} \frac{b(v,q)}{\|v\|_V} \geq \beta\|q\|_Q.$$

□

14.2.2 Finite element discretization

In the discretization of Stokes equations, we need a pair of finite element subspaces $V_h \subset V$ and $Q_h \subset Q$.

$$\begin{cases} a(u_h, v_h) + b(v_h, p_h) = (f, v_h) & \forall\, v_h \in V_h, \\ b(u_h, q_h) = 0 & \forall\, q_h \in Q_h. \end{cases} \tag{14.37}$$

Let
$$B = -\mathrm{div}_h : V_h \mapsto Q_h, \quad B^t = \nabla : Q_h \mapsto (H^{-1}(\Omega))^n.$$
Then the Stokes equation can be written in a matrix form

$$\begin{pmatrix} A_h & B_h^t \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} f_h \\ 0 \end{pmatrix}. \tag{14.38}$$

As in the continuous case, the discrete Stokes equation is well-posed if the inf-sup condition is satisfied

$$\inf_{q_h\in Q_h} \sup_{v_h\in V_h} \frac{b(v_h, q_h)}{|v_h|_1 |q_h|_0} \geq \alpha > 0. \tag{14.39}$$

If we can choose α independently of mesh size h, we then call the pair $V_h \times Q_h$ to be a stable element. In this case, the discrete Stokes equation is well-posed, in certain sense, uniformly with respect to the mesh size h.

One immediate consequence of the above inf-sup condition is the following best approximation property.

Lemma 105. *Let $u \in V$ be divergence-free:* $\mathrm{div} u = 0$. *Then, under the inf-sup condition* (14.39)

$$\inf_{w_h\in V_h(0)} |u - w_h|_1 \leq (1 + \frac{1}{\alpha}) \inf_{v_h\in V_h} |u - v_h|_1.$$

Here $V_h(0)$ is the discrete divergence-free space:

$$V_h(0) = \{v_h \in V_h : \ b(v_h, q_h) = 0 \quad \forall\, q_h \in Q_h.$$

Proof. Given $v_h \in V_h$, let $r_h \in V_h$ be such that

$$b(r_h, q_h) = b(u - v_h, q_h) \quad \forall\ q_h \in Q_h.$$

□

Theorem 105. *For a pair of stable element $V_h \times Q_h$, namely the inf-sup condition* (14.36) *is satisfied uniformly with respect to h, then*

$$|u - u_h|_1 + |p - p_h|_0 \le C(\sup_{v_h \in V_h} |u - v_h|_1 + \inf_{q_h \in Q_h} |p - q_h|_0) \tag{14.40}$$

for some constant C independent of h.

Proof. The proof is simple □

Exercise 1. Prove the best approximation property (14.40) holds uniformly with respect to h (if and) only if the inf-sup condition (14.39) holds uniformly with respect to h.

The verification of the discrete inf-sup condition is not an easy task in general. Let us describe one convenient technique in this direction.

Lemma 106. *The discrete inf-sup condition is uniformly satisfied if there is a linear operator $\Pi_h : V \mapsto V_h$ such that*

$$b(v - \Pi_h v, q_h) = 0 \quad \forall\ q_h \in Q_h \tag{14.41}$$

and

$$|\Pi_h v|_1 \le c|v|_1 \quad \forall\ v \in V \tag{14.42}$$

for some constant c independent of h.

The proof of this lemma is straightforward by using the inf-sup condition (14.36) on the continuous level.

Exercise 2. Prove Lemma 106.

$P_2 - P_0$ elements and Crouzeix-Raviart elements

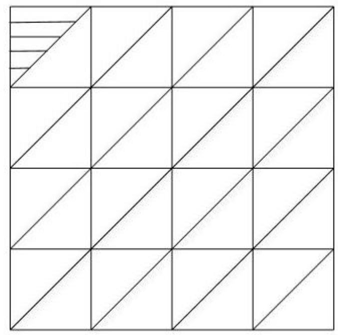
One seemingly natural pair of elements is continuous piecewise P_1 (for velocity) versus piecewise constant (for pressure). For the mesh in Fig 2, let n be the number of nodes on x-axis, then the number of interior vertices is $n_v = (n-2)^2$ and the number of elements is $n_t = 2(n-1)^2$. We know that $\dim V_h \le 2n_v = 2(n-2)^2$ and $\dim Q_h = n_t - 1 = 2(n-1)^2 - 1$. So $\dim V_h < \dim Q_h$, the discrete system can not be well defined.

Moreover, if $u = (u_1, u_2)^T \in V_h$ and $\operatorname{div} u = 0$, meaning $\partial_x u_1 + \partial_y u_2 = 0$. Consider the marked element in Fig 2, $u_2 = 0$ on the vertical edge, then $\partial_x u_1 = 0$ on the vertical edge. Then we have $\partial_x u_1 = 0$ in the element since u is a linear polynomial. Since $u_1 = 0$ on the boundary, then $u_1 = 0$ in the element, thus $\partial_y u_2 = 0$ in the element. Since $u_2 = 0$ on the boundary, then $u_2 = 0$ in the element, thus $u \equiv 0$. It means that for some special grid, the only divergence free function is the zero function, hence the inf-sup condition can not be satisfied.

To fix this problem, we can enrich the velocity space. We shall now discuss two such enrichments. With the piecewise constant functions still used for the pressure space, the first one is to use P_2 element for velocity and the second one is to use nonconforming P_1 element for velocity (Crouzeix-Raviart element).

We shall now prove that the $P_2 - P_0$ is indeed a stable element. We proceed to verify the conditions (14.41) and (14.42). For any $v \in V$, we need to define $v_h = \Pi_h v$ satisfying (14.41) which, in the present situation, reads, for each element T

$$\int_T \operatorname{div}(v - v_h) = \int_{\partial T} (v - v_h) \cdot n = 0.$$



This last condition can obviously be satisfied if we construct v_h in such a way that, for each edge e_i of T,

$$\int_{e_i} v_h = \int_{e_i} v, \quad i = 1, 2, 3. \tag{14.43}$$

The above condition can be easily satisfied by defining the value of v_h at the midpoint of each e_i using the relation (14.43). But since we are in P_2 we also need to define v_h on the nodal points as well. The procedure we will take is first to define the nodal values of v_h and then define the midpoint values of v_h by (14.43). To do this we use make use the Scott-Zhang interpolantion introduced in §4.4 which we denote here by Π_1. We then define

$$\Pi_h v = \Pi_1 v + \Pi_2(v - \Pi_1 v) \tag{14.44}$$

where $\Pi_2 : V \mapsto V_h$ is defined locally such that, for each $v \in V$, $\Pi_2 v$ is zero on each vertex and its value of the midpoint of any edge e is determined by a relation like (14.43)

$$\int_e \Pi_2 v = \int_e v.$$

Then for any $q_h \in Q_h$,

$$\begin{aligned} b(v - \Pi_h v, q_h) &= \sum_K \int_K \operatorname{div}(v - \Pi_h v) q_h = \sum_K \int_{\partial K} (v - \Pi_h v) \cdot n q_h \\ &= \sum_K \int_{\partial K} (I - \Pi_2)(v - \Pi_1 v) \cdot n q_h = 0. \end{aligned} \tag{14.45}$$

Meanwhile, by the properties of the Scott-Zhang interpolation in §4.4,

$$\begin{aligned} |\Pi_h v|_1 &\le |\Pi_1 v|_1 + |\Pi_2(v - \Pi_1 v)|_1 \\ &\le |v|_1 + \sum_K (h_K^{-1} \|v - \Pi_1 v\|_{0,\omega_K} + |v - \Pi_1 v|_{1,\omega_K} \lesssim |v|_1 \end{aligned} \tag{14.46}$$

Thus, the interpolation Π_h constructed in this way satisfies all the desired properties.

Exercise 3. Prove that the Π_h defined by (14.44) satisfies both (14.41) and (14.42).

As we see from the above analysis, as far as inf-sup condition is concerned, the digree of freedoms on all the nodal points in the P_2 element is not necessary. So let us just throw them away! The resulting element is then a nonconforming piecewise linear element which is continuous at all midpoints. This element is known as the Crouzeix-Raviart element. Indeed, for purely Dirichlet boundary condition, inf-sup condition can be verified easily. But, unfortunately, this element seems to only work for the pure Dirichet boundary condition for sure since the discrete Korn's inequality may fail to hold for other boundary conditions. Since the Stokes equation with a purely Dirichet boundary condition is of very little practical importance, the significance of this pair of element is then very limited.

14.2.3 Some examples of stable elements

In this section, we describe some examples of stable elements.

Example 9. Here we have some examples in 2-dimension that are commonly used.

1. Natural but divergent elements: $\mathbb{V}_h = (\mathcal{P}_1^0)^2$, $\mathbb{Q}_h = \mathcal{P}_0^{-1}$.
2. Stable elements: $\mathbb{V}_h = (\mathcal{P}_2^0)^2$, $\mathbb{Q}_h = \mathcal{P}_0^{-1}$.
3. Mini element: $\mathbb{V}_h = (\mathcal{P}_1^0 \oplus \text{Bubble})^2$, $\mathbb{Q}_h = \mathcal{P}_1^0$.
4. More unstable elements:
   - $\mathbb{V}_h = (\mathcal{P}_2^0)^2$, $\mathbb{Q}_h = \mathcal{P}_1^{-1}$
   - $\mathbb{V}_h = (\mathcal{P}_3^0)^2$, $\mathbb{Q}_h = \mathcal{P}_2^{-1}$
5. Stable elements:
   - $\mathbb{V}_h = (\mathcal{P}_k^0)^2$, $\mathbb{Q}_h = \mathcal{P}_{k-1}^{-1}$ with $k \geq 4$. This is Scott-Vogelius element. It is stable provided that there is no singular vertex in the mesh.
   - $\mathbb{V}_h = (\mathcal{P}_2^0)^2$, $\mathbb{Q}_h = \mathcal{P}_1^0$. This is called Taylor-Hood element.

No bubbles

Taylor-Hood elements

These elements refer to the following

- P_2^0–P_1^0 for both $\mathbb{R}^2$ (triangle) and $\mathbb{R}^3$ (tedrahedral).
- P_3^0–P_2^0 for both $\mathbb{R}^2$ (triangle) and $\mathbb{R}^3$ (tedrahedral).
- Q_2^0–Q_1^0 for $\mathbb{R}^2$ (square).

Scott-Vogelius elements

It is known that the $P_k^0 - -P_{k-1}^{-1}$ are not stable for $k \leq 3$. But Scott-Vogelius proved that

- P_k^0–P_{k-1}^{-1} for $\mathbb{R}^2$ (triangle) and $k \geq 4$

is indeed stable for triangular grid with mild restrictions.

Stablization by bubble functions

A bubble function is a function that is defined locally on each element and has support on that element. The most commonly used bubble function is the cubic bubble

$$b_3(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x).$$

MINI-element

The MINI-element is the known stable element with the minimal degree of freedom and it consists of

- $P_1^0 + \{b_3\} - P_1^0$

Crouzeix-Raviart elements

To stablize the $P_2^0 - P_1^{-1}$ element, Crouzeix and Raviart added a bubble to the velocity space:

- $P_2^0 + \{b_3\} - P_1^{-1}$.

Comments on divergence-free conditions

From the divergence-free condition $\mathrm{div} u = 0$, we can derive $\int_{\partial\Omega} u \cdot n = 0$.

1. If the pressure space is discontinuous, then on each element K, we have $\int_{\partial K} u_h \cdot n = 0$.
2. For Scott-Vogelius element, we actually have $\mathrm{div} u_h = 0$ everywhere, which is equivalent to $\int_G u_h \cdot n = 0$, $\forall$ open set $G \subset \Omega$.

14.3 Necas inequality, Korn inequality and inf-sup condition

We want the discretization to be well-posed. The inf-sup condition for $a(\cdot,\cdot)$ and $c(\cdot,\cdot)$ on K and H are trivial. We only need to worry about the inf-sup condition for $b(\cdot,\cdot)$, namely the inf-sup condition for $B = -\mathrm{div}$, which reads

$$\inf_{q_h \in Q_h} \sup_{\mathbf{v_h} \in \mathbb{V}_h} \frac{(q_h, div \mathbf{v}_h)}{\|q_h\|_{0,\Omega} |\mathbf{v}_h|_{1,\Omega}} \geq \beta. \tag{14.47}$$

The following inequality was first proved by Necas.

Theorem 106 (Necas's inequality). *Let $\Omega \in \mathbb{R}^d$ be a connected domain with a Lipschitz boundary, then there exists a constant C such that for all $p \in L^2(\Omega)$,*

$$\|p\|_{L^2} \leq C \left(\|p\|_{H^{-1}}^2 + \sum_{i=1}^d \|\frac{\partial p}{\partial x_i}\|_{H^{-1}}^2 \right)^{1/2} \tag{14.48}$$

Proof. A simple proof for the general cases can be found in [**?**] by J. Bramble. Here we only present a special case that $\Omega = \mathbb{R}^n$ by using Fourier transform. Let $\hat{p}(\xi) = \mathscr{F}(p)$ be the Fourier transform of p, then

$$\|p\|_{-1}^2 = \|\frac{1}{\sqrt{1+|\xi|^2}} \hat{p}\|_{\mathbb{R}^d}^2, \quad \|\frac{\partial p}{\partial x_i}\|_{-1}^2 = \|\frac{\xi_i}{\sqrt{1+|\xi|^2}} \hat{p}\|_{\mathbb{R}^d}^2,$$

so in this case

$$\|p\|_{L^2}^2 = \|p\|_{-1}^2 + \sum_{i=1}^d \|\frac{\partial p}{\partial x_i}\|_{-1}^2.$$

□

The following Korn's inequality can be proved directly from the Necas's inequality.

Theorem 107 (Korn's inequality). *Let $\Omega \in \mathbb{R}^d$ be a conneced domain with a Lipschitz boundary, then there exists a constant C such that for all $u \in [H^1(\Omega)]^d$,*

$$\|u\|_{H^1} \leq C \left(\|u\|_{L^2}^2 + \|\epsilon(u)\|_{L^2}^2 \right)^{1/2} \tag{14.49}$$

Proof. We only consider the smooth vectors u since smooth functions are dense in both $L^2(\Omega)$ and $H^1(\Omega)$. Take $p = \frac{\partial u_i}{\partial x_j}$ in (14.48), then

$$\|\frac{\partial u_i}{\partial x_j}\|_{L^2} \le \|\frac{\partial u_i}{\partial x_j}\|_{H^{-1}} + \sum_{k=1}^{d} \|\frac{\partial^2 u_i}{\partial x_j \partial x_k}\|_{H^{-1}}.$$

Notice that

$$\|\frac{\partial v}{\partial x_k}\|_{H^{-1}} \le \|v\|_{L^2}, \quad \forall v \in L^2$$

$$\frac{\partial^2 u_i}{\partial x_j \partial x_k} = \frac{\partial \varepsilon_{ik}}{\partial x_j} + \frac{\partial \varepsilon_{ij}}{\partial x_k} - \frac{\partial \varepsilon_{jk}}{\partial x_i}.$$

Then we have the desired result. □

At the end of this section, we will prove the inf-sup condition for $B = \mathrm{div} : [H_0^1]^d \mapsto (L_0^2)'$ with the help of Necas inequality. Here $L_0^2(\Omega) \triangleq \{p \in L^2(\Omega) | \int_\Omega p \mathrm{d}x = 0\}$. We hope to prove that

$$\inf_{q \in L_0^2} \sup_{u \in (H_0^1)^d} \frac{(\mathrm{div} u, q)}{\|q\|_{L^2} \|u\|_{(H^1)^d}} = \beta > 0. \tag{14.50}$$

The proof is divided into several steps.

Lemma 107. *Under the same assumption of domain as Theorem 106, the Necas inequality* (14.48) *is equivalent to*

$$\|p\|_{L^2} \le C \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{H^{-1}}, \quad \forall p \in L_0^2(\Omega). \tag{14.51}$$

Proof. If (14.51) holds, then $\forall p \in L^2$, let

$$p = \bar{p} + p_0, \quad p_0 \in L_0^2, \bar{p} \in \mathbb{R}.$$

Then we have

$$\begin{aligned} \|p_0\|_{L^2} + \|p\|_{-1} &= \sup_{v \in L^2} \frac{(-p_0, v)}{\|v\|_{L^2}} + \sup_{v \in H_0^1} \frac{(p, v)}{\|v\|_1} \\ &\gtrsim \sup_{v \in H_0^1} \frac{(-p_0, v)}{\|v\|_1} + \sup_{v \in H_0^1} \frac{(p, v)}{\|v\|_1} \ge \sup_{v \in H_0^1} \frac{(\bar{p}, v)}{\|v\|_1} = \|\bar{p}\|_{-1}, \end{aligned}$$

Notice that $\|\bar{p}\|_{L^2} = \frac{\|1\|_{L^2}}{\|1\|_{-1}} \|\bar{p}\|_{-1} \simeq \|\bar{p}\|_{-1}$, then

$$\begin{aligned} \|p\|_{L^2}^2 &= \|p_0\|_{L^2}^2 + \|\bar{p}\|_{L^2}^2 \lesssim \|p_0\|_{L^2}^2 + \|\bar{p}\|_{-1}^2 \\ &\lesssim \|p_0\|_{L^2}^2 + \|p\|_{-1}^2 \\ &\lesssim \|p\|_{-1}^2 + \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{-1}^2. \end{aligned}$$

On the other hand, we only need to prove that

$$\|p\|_{-1} \lesssim \sum_{i=1}^{d} \|\frac{\partial p}{\partial x_i}\|_{-1}, \quad \forall p \in L_0^2.$$

This inequality follows from the standard contradiction argument. If it is not true, then there exist a sequence $\{p_n \in L_0^2\}$ such that $\|p_n\|_{-1} = 1$ and $\sum_{i=1}^d \|\frac{\partial p_n}{\partial x_i}\|_{-1} \to 0$. From Necas inequality (14.48), we immediately see that $\{p_n\}$ is a uniformly bounded sequence in L^2. Since the embedding of H^1 into L^2 is compact, thus the adjoint embedding of L^2 into H^{-1} is also compact. Then there exists a subsequence of $\{p_n\}$ that convergences to q in H^{-1} (Here we still represent the subsequence by $\{p_n\}$). Then

$$\|p_n - p_m\|_{L^2} \lesssim \|p_n - p_m\|_{-1} + \sum_{i=1}^d \|\frac{\partial(p_n - p_m)}{\partial x_i}\|_{-1} \to 0, \quad \forall n, m \to \infty$$

which means $\{p_n\}$ is Cauchy sequence in L^2, thus $p_n \to \tilde{q}$ in L_0^2. Since $p_n \to q$ in H^{-1} then $q = \tilde{q} \in L_0^2$. Then we have

$$\sum_{i=1}^d \|\frac{\partial q}{\partial x_i}\|_{-1} = \lim_{n\to\infty} \|\frac{\partial p_n}{\partial x_i}\| = 0,$$

which implies q is a constant function. From $p_n \to q$ in L^2 and $p_n \in L_0^2$ we immediately have $q = 0$, which contradicts to the fact that

$$\|q\|_{L^2} = \lim_{n\to\infty} \|p_n\|_{L^2} = 1.$$

□

Lemma 108. *The inf-sup condition* (14.50) *is equivalent to* (14.51).

Proof. If (14.51) holds, by definition we have

$$\|q\|_{L^2} \lesssim \sum_{i=1}^d \sup_{v\in H_0^1} \frac{(q, \frac{\partial v}{\partial x_i})}{\|v\|_1}, \quad \forall q \in L_0^2.$$

Then there exists an index $1 \le j \le d$, such that

$$\|q\|_{L^2} \lesssim \sup_{v\in H_0^1} \frac{(q, \frac{\partial v}{\partial x_j})}{\|v\|_1},$$

thus

$$\|q\|_{L^2} \lesssim \sup_{v\in H_0^1} \frac{(q, \frac{\partial v}{\partial x_j})}{\|v\|_1} = \sup_{u=ve_j\in(H_0^1)^d} \frac{(q, \mathrm{div} u)}{\|u\|_1} \le \sup_{u\in(H_0^1)^d} \frac{(q, \mathrm{div} u)}{\|u\|_1}.$$

On the other hand, if (14.50) is true, then for all $q \in L_0^2$

$$\|q\|_{L^2} \lesssim \sup_{u\in(H_0^1)^d} \frac{(q, \mathrm{div} u)}{\|u\|_1} = \sup_{u=(u_1,\cdots,u_d)\in(H_0^1)^d} \sum_{i=1}^d \frac{(q, \frac{\partial u_i}{\partial x_i})}{\|u\|_1} \lesssim \sup_{u=(u_1,\cdots,u_d)\in(H_0^1)^d} \sum_{i=1}^d \frac{(q, \frac{\partial u_i}{\partial x_i})}{\|u_i\|_1}$$

which finishes the proof. □

In summary, we have proved that Necas's inequality, (14.51) and inf-sup condition (14.50) are equivalent. Thus, Necas's inequality is of fundamental importance while its proof is not elementary.

For the non-homogeneous boundary condition, let us prove the more general inf-sup condition based on Necas inequality.

Theorem 108 (General Inf-sup condition). *Let $\Omega \in \mathbb{R}^d$ be a connected domain with a Lipschitz boundary. For the Banach space $V \supset (H_0^1)^d$,*

1. *If $V \subset \{u \mid \int_{\partial\Omega} u \cdot n = 0\}$, then the inf-sup condition*

$$\inf_{q\in Q} \sup_{u\in V\cap(H^1)^d} \frac{(\mathrm{div}, q)}{\|q\|_0\|u\|_1} = \beta > 0 \tag{14.52}$$

holds for $Q = L_0^2$.

2. *If $V \not\subset \{u \mid \int_{\partial\Omega} u \cdot n = 0\ \Omega\}$, then the inf-sup condition* (14.52) *holds for $Q = L^2$.*

Proof. Using the standard contradiction argument, if (14.52) fails, there exists a sequence $\{q_n \in Q\}$ such that

$$\|q_n\|_0 = 1, \quad \sup_{u\in V} \frac{(\mathrm{div}, q_n)}{\|u\|_1} \to 0.$$

Since $V \supset (H_0^1)^d$,

$$\sup_{u\in V} \frac{(\mathrm{div}, q_n)}{\|u\|_1} \le \sup_{u\in(H_0^1)^d} \frac{(\mathrm{div}, q_n)}{\|u\|_1} \to 0.$$

By Lemma 108, we obtain

$$\sum_{i=1}^d \|\frac{\partial q_n}{\partial x_i}\|_{-1} \to 0.$$

Using the compact embedding of L^2 in H^{-1}, there is a sequence, call it again $\{q_n\}$, such that $q_n \to p$ in H^{-1}. By Necas inequality (14.48),

$$\|q_n - q_m\|_0 \lesssim \|q_n - q_m\|_{-1} + \sum_{i=1}^d \|\frac{\partial(q_n - q_m)}{\partial x_i}\|_{-1} \to 0, \quad \forall n, m \to \infty.$$

which means $\{q_n\}$ is Cauchy sequence in L^2, thus $q_n \to p$ in L^2. Then we have

$$\sum_{i=1}^d \|\frac{\partial p}{\partial x_i}\|_{-1} = \lim_{n\to\infty} \|\frac{\partial q_n}{\partial x_i}\|_{-1} = 0,$$

which implies p is constant function. For the case 1, $Q = L_0^2$ means $p = 0$, which contradicts to

$$\|p\|_0 = \lim_{n\to\infty} \|q_n\|_0 = 1.$$

For the case 2, we see that

$$(\mathrm{div} u, p) = \int_{\partial\Omega} pu \cdot n,$$

since $V \not\subset \{u \mid \int_{\partial\Omega} u \cdot n = 0\}$, then there exists $u^* \in V$, such

$$\frac{(\mathrm{div} u^*, p)}{\|u^*\|_1} > 0.$$

This is a contradiction since

$$0 < \frac{(\mathrm{div} u^*, p)}{\|u^*\|_1} = \lim_{n\to\infty} \frac{(\mathrm{div} u^*, q_n)}{\|u^*\|_1} \le \lim_{n\to\infty} \sup_{u\in V} \frac{(\mathrm{div} u, q_n)}{\|u\|_1} = 0.$$

Thus the theorem is proved. □

The following corollary can be proved by the similar contradiction argument and the compactness of H^1 in L^2, we leave the proof to readers.

Corollary 10. *Assume that* $\mathrm{meas}(\Gamma_D) \neq 0$*, then*

$$\|u\|_1 \lesssim \|\epsilon(u)\|_0, \quad \forall u \in (H_D^1)^d. \tag{14.53}$$

14.4 A kernel space projection method

For finite difference method, staggered grid in Figure 14.1 is usually used. Staggered finite differences means that the x and y component of velocity and pressure are defined at physically different grid points, with pressure p at the cell centre and the velocity components at the centre of the cell faces.. The main benefit of the staggered gird over collocated grid, is that, unlike the collocated grid, the staggered grid does not results in the decoupling of the pressure and velocity, leading to the checkerboard problem.

From the last section, we observe the key role of the kernel space. In this section, we introduce the projection method for (time dependent) Navier-Stokes equation.

$$\begin{cases} \rho(\partial_t u + (u \cdot \nabla)u) - \nu \Delta u + \nabla p = f \\ \operatorname{div} u = 0 \end{cases}$$

The variational form is

$$\begin{cases} \rho(\partial_t u + (u \cdot \nabla)u, v) - \nu(\nabla u, \nabla v) - (\operatorname{div} v, p) = (f, v) \\ (\operatorname{div} u, q) = 0 \end{cases}$$

$u \in V_0 \equiv Ker(\operatorname{div}) = \{v \in H^1 : \operatorname{div} v = 0\}$.

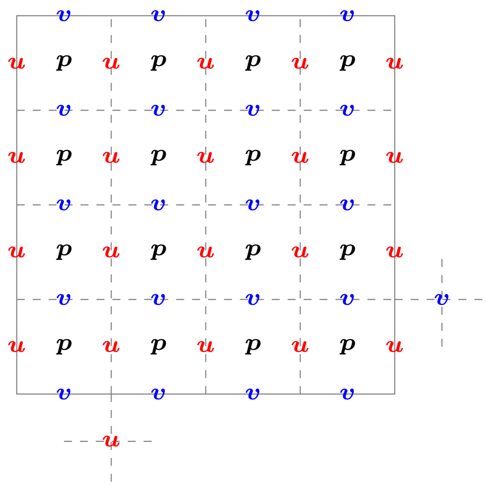


Fig. 14.1. Staggered Grid

The projection method which is first developed by Chorin [**?**] and Temam [**?**] in 1960s. The method can be written as following.

Step 1: By moving the nonlinear term to right hand side, the momentum equation can be written as

$$\rho \partial_t u - \nu \Delta u = -\rho(u \cdot \nabla)u + f$$

By time discretization, we will solve in $Ker(\operatorname{div})$,

$$\rho \frac{u^* - u^n}{\Delta t} - \nu \Delta u^* = f^n \equiv -\rho(u^n \cdot \nabla)u^n + f. \tag{14.54}$$

The problem is that we cannot make sure $u^* \in V_0$, namely $\operatorname{div} u^* \neq 0$.

Step 2: Project u^* to divergence free space $V_0 = Ker(\text{div})$.

The following well-known L^2-orthogonal Helmholtz decomposition (or Hodge decomposition):

$$u^* = u_0 + \omega, \text{ where } u_0 \in V_0, u_0 = \text{curl}\psi, \omega = \nabla\phi.$$

So, $u^* - u_0 = \nabla\phi$. Since $\rho\partial_t u \sim \nabla p$, by introducing a proper scaling, we have

$$\rho\frac{u^* - u_0}{\Delta t} = \nabla\phi.$$

$$u^{n+1} = u_0 := \text{ the projection of } u^* \text{ into } V_0.$$

Question: Is this projection process computationally feasible?

Since

$$\rho\frac{u^* - u^{n+1}}{\Delta t} = \nabla\phi. \tag{14.55}$$

and

$$\text{div}u^{n+1} = 0. \tag{14.56}$$

Taking the divergence on both side of equation (14.55), we have

$$\begin{cases} \rho\dfrac{\text{div}u^*}{\Delta t} = \Delta\phi \\ \dfrac{\partial\phi}{\partial n} = 0 \end{cases} \tag{14.57}$$

We can solve ϕ by equation (14.57) which is equivalent to equation (14.56). The key point is how to choose a proper boundary condition for equation (14.57). After solving ϕ, u^{n+1} can be updated by (14.55):

$$u^{n+1} = u^* - \rho^{-1}\Delta t\nabla\phi.$$

Step 3: Find p^{n+1}. For the traditional projection method, they will choose $p^{n+1} = \phi$. However, we can do better to make it satisfies

$$\begin{cases} \rho\dfrac{u^{n+1} - u^n}{\Delta t} - \nu\Delta u^{n+1} + \nabla p^{n+1} = f^n \\ \text{div}u^{n+1} = 0 \end{cases} \tag{14.58}$$

Using (14.58) - (14.54),

$$\rho\frac{u^{n+1} - u^*}{\Delta t} - \nu\Delta(u^{n+1} - u^*) + \nabla p^{n+1} = 0$$

By (14.55),

$$-\nu\Delta(u^{n+1} - u^*) + \nabla(p^{n+1} - \phi) = 0$$

So,

$$-\nu\Delta(\rho^{-1}\Delta t\nabla\phi) + \nabla(p^{n+1} - \phi) = 0$$

Assume that $\nabla\Delta = \Delta\nabla$, then

$$\nabla(-\nu\rho^{-1}\Delta t\Delta\phi + p^{n+1} - \phi) = 0$$

So,

$$p^{n+1} = \phi + \nu\rho^{-1}\Delta t\Delta\phi.$$

We finally remark that the key step is the step of projection, which is a divergence correction process.