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Appendix: Differential forms and a unified theory of exact sequences

18.1 A unified formulation of *grad*, *curl* and *div*

Before discussing about differential forms, we will introduce a unified formulation of *grad*, *curl* and *div* first. Let's look at the definition of *grad*, *curl* and *div*. Consider a bounded domain $\Omega \in \mathbb{R}^3$. For a scalar function

$$u : \Omega \to \mathbb{R}^1.$$

$$\text{grad}u = \begin{pmatrix} \dfrac{\partial u}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} \\ \dfrac{\partial u}{\partial x_3} \end{pmatrix}, \quad \text{grad} = \begin{pmatrix} \dfrac{\partial}{\partial x_1} \\ \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_3} \end{pmatrix} = \nabla.$$

Vector function $\mathbf{u} = (u_1, u_2, u_3)$.

$$\textbf{curl } \mathbf{u} = \begin{pmatrix} \dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3} \\ \dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1} \\ \dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2} \end{pmatrix} = \nabla \times \mathbf{u}$$

$$\text{d}iv\mathbf{u} = \sum_{i=1}^{3} \partial_i u_i = \nabla \cdot \mathbf{u}.$$

curl grad: $\nabla \times (\nabla u) = 0$. div grad: $\nabla \cdot (\nabla \times \mathbf{u}) = 0$. So, if ω does not have holes,

$$Range(grad) \subset Ker(curl), \quad Range(curl) \subset Ker(div).$$

Define

$$H(D; \omega) = \{v \in L^2(\Omega) | Dv \in L^2(\Omega)\},$$

where $D = grad, curl, div$. We have the exact sequence:

$$R \longrightarrow H(\text{grad}) \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$$

Exact sequence: If the range of any operator in the sequence is equal to the kernel of the operator on the right.

The exact sequence is one of the most important properties, as the kernel of the three operators then each has an exact description.

18.1.1 *k*-vector, Hodge star, exterior derivatives

Here we give intuitive explanation on differential forms rather than giving accurate mathematical definitions. Any vector $\xi \in \mathbb{R}^3$ is uniquely determined by magnititude and direction. We call such vector 1-vector.

Similarly, we can define 2-vector. Let's look at a special case first. Assume that

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then $\mathbf{e_i} \wedge \mathbf{e_j}$ $(i = 1, 2, 3;\ j = 1, 2, 3;\ i \neq j)$ forms a 2-vector. And it satisfies:

$$\mathbf{e_i} \wedge \mathbf{e_j} = -\mathbf{e_j} \wedge \mathbf{e_i}$$
$$\mathbf{e_i} \wedge \mathbf{e_i} = 0$$

In general, a 2-vector is formed by two 1-vectors, which is also charaterized by its magnitude and direction. For example, $\mathbf{u} \wedge \mathbf{v}$ is a 2-vector, whose magnitude is the area of parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, and whose direction is determined by orientation. Assume that

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e_i},\ \mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e_j}$$

Therefore,

$$\mathbf{u} \wedge \mathbf{v} = (\sum_{i=1}^{3} u_i \mathbf{e_i}) \wedge (\sum_{j=1}^{3} v_j \mathbf{e_j}) = \sum_{i,j} u_i v_j \mathbf{e_i} \wedge \mathbf{e_j}$$

The case of 3-vector is quite similar to 2-vector. For example, $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ is a 3-vector, whose magnitude is the volume of parallelpiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and whose direction is determined by orientation. Assume that

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e_i},\ \mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e_j},\ \mathbf{w} = \sum_{k=1}^{3} w_j \mathbf{e_k}$$

Therefore,

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = (\sum_{i=1}^{3} u_i \mathbf{e_i}) \wedge (\sum_{j=1}^{3} v_j \mathbf{e_j}) \wedge (\sum_{k=1}^{3} w_j \mathbf{e_k}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$$

There is something interesting with determinant. In fact, determinant is defined via wedge product in a similar way to 3-vector in $\mathbb{R}^3$, i.e. assume that matrix $A = (\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n})$, then:

$$\det(A) = \det(\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}) = *(\mathbf{a_1} \wedge \mathbf{a_2} \wedge \cdots \wedge \mathbf{a_n})$$

After learning differential forms, we can define exterior derivative acting on differential forms. We still work on $\mathbb{R}^3$ in this section. And we will begin with 0-vector.

0-vector is the so-called 0-form. Assume that f is f is 0-form then f is a map from $\mathbb{R}^3$ to $\mathbb{R}$, and

$$d_0 f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \mathbf{e_i} = \sum_{i=1}^{3} (\nabla f)_i \mathbf{e_i}$$

With the definition of exterior derivation for 0-form, we can define exterior derivative for higher order differential forms recursively. We will elaborate those cases one by one. For a 1-form $\mathbf{u} = \sum_{i=1}^{3} u_i(x)\mathbf{e_i}$, exterior derivative d_1 is defined as:

$$d_1\mathbf{u} = \sum_{i=1}^{3} d_1(u_i\mathbf{e_i}) = \sum_{i=1}^{3}(d_0u_i) \wedge \mathbf{e_i}$$

Notice that $d_0u_i = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\mathbf{e_j}$, so

$$\begin{aligned} d_1\mathbf{u} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\mathbf{e_j} \wedge \mathbf{e_i} \\ &=(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3})\mathbf{e_2} \wedge \mathbf{e_3} + (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1})\mathbf{e_3} \wedge \mathbf{e_1} + (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})\mathbf{e_1} \wedge \mathbf{e_2} \\ &= \sum_{i=1}^{3}(\nabla \times \mathbf{u})_i(*\mathbf{e_i}) \end{aligned}$$

Similarly, for a 2-vector $\mathbf{u} = \sum_{i=1}^{3} u_i(*\mathbf{e_i})$, we have:

$$\begin{aligned} d_2\mathbf{u} &= \sum_{i=1}^{3}(d_0u_i) \wedge (*\mathbf{e_i}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\mathbf{e_i} \wedge (*\mathbf{e_i}) \\ &= \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3} = \sum_{i=1}^{3}(\nabla \cdot \mathbf{u})\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3} \end{aligned}$$

Now we will give a brief summary about differential k-form in $\mathbb{R}^3$ and the corresponding exterior derivatives in table 19.1.

Table 18.1. Differential k-form in $\mathbb{R}^3$ and exterior derivative d_k

k-form		proxy of d_k
0	$u(x)$	*grad*
1	$u_1(x)\mathbf{e_1} + u_2(x)\mathbf{e_2} + u_3(x)\mathbf{e_3}$	*curl*
2	$u_1(x)\mathbf{e_2} \wedge \mathbf{e_3} + u_2(x)\mathbf{e_3} \wedge \mathbf{e_1} + u_3(x)\mathbf{e_1} \wedge \mathbf{e_2}$	*div*
3	$u(x)\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$	

An important property of exterior derivative is $d_{i+1}d_i = 0$. It means that in $\mathbb{R}^3$:

$$\nabla \times (\nabla u) = 0,\ \nabla \cdot (\nabla \times \mathbf{u}) = 0$$

This property is the so-called Poincare theorem.

18.2 Stokes theorem

In this section we will discuss about Stokes theorem, which is an important theorem about differential forms. A unified representation of Stokes theorem is

$$\int_{M_k} d_{k-1}\omega = \int_{\partial M_k} \omega$$

where M_k is an arbitrary k-manifold, ∂ is boundary operator, ω is a $(k-1)$-form. Next we will take $\mathbb{R}^3$ as example to see the concrete expressions of Stokes theorem. With respect to the operators, the specific form is:

1. d = grad, M = C (circle),
$$\int\limits_C \nabla u \cdot ds = \int\limits_{\partial C} u$$
2. d = curl, M = S (surface),
$$\iint\limits_S (\nabla \times \underset{\sim}{v}) ds = \int\limits_{\partial S} \underset{\sim}{v} \cdot \mathbf{t}$$
3. d = div, M = Ω (volume),
$$\iiint\limits_\Omega \nabla \cdot \underset{\sim}{v} = \iint\limits_{\partial\Omega} \underset{\sim}{v} \cdot \mathbf{n}$$

18.3 A unified theory of $H(grad)$, $H(curl)$ and $H(div)$ finite elements

Before discussing about finite element method, we define Sobolev space for differential k-form first. Recall the definition of Hilbert space for k-form:

$$H(d_k;\Omega) = \{v \in L^2(\Omega) : \ d_k v \in L^2(\Omega)\}$$

where d_k is the exterior derivative. In $\mathbb{R}^3$, the Sobolev spaces for $k = 0, 1, 2, 3$ are the widely used Hilbert spaces $H(grad)$, $H(curl)$, $H(div)$ and L^2.

Next, we will give a formal definition of finite element space. A finite element method is defined by a triple $(K, \Sigma_K, \mathcal{P}_K)$:

- K is a simplex in $\mathbb{R}^d$, where $d = 2$ or 3.
- Σ_K is a set of degrees of freedom (or saying that Σ_K is a set of linear functionals).
- $\mathcal{P}_K$ is a space of polynomials on K s.t. Σ_K gives its dual basis.

Example This is an $1D$ example. Assume that:

- $K = [a, a+h]$ for a given $a \in \mathbb{R}$.
- $\Sigma_K = \{u(a), u(a+h)\}$.
- $\mathcal{P}_K = \{\textit{linear polynomials}\}$.

then we know that $v \in \mathcal{P}_K$ looks like $v = \alpha + \beta x$.

18.3.1 The definitions

For $k = 0$, the $H_h^1(\Omega)$ is continuous piecewise linear finite element spaces and $L_h^2(\Omega)$ is the piecewise constant finite element space.

The space H_h^{curl}: For H_h^{curl}, on each element K, the shape function space is 6-dimensional

$$\mathcal{P} = \{\alpha + \beta \times x : \alpha, \beta \in \mathbb{R}^3\}$$

and the degrees of freedoms are tangential integral on each element edge

$$\mathcal{N} = \left\{ \int_e v \cdot t : \text{for each edge } e \subset K \right\}.$$

The space H_h^{div}: For $k = 0$ the shape function space for H_h^{div} on each element K is 4-dimensional

$$\mathcal{P} = \{\alpha + \beta x : \alpha \in \mathbb{R}^3, \beta \in \mathbb{R}\}$$

and the degrees of freedoms are normal integral on each element face

$$\mathcal{N} = \left\{ \iint_F v \cdot n : \text{ for each face } F \subset K \right\}.$$

The finite elements are summarised in the figure below:

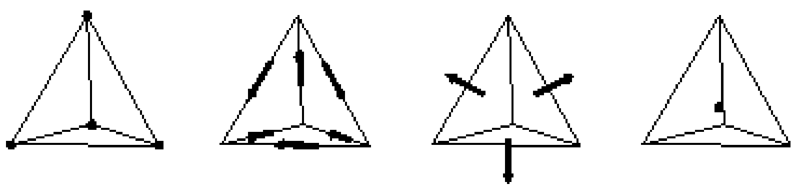

Fig. 18.1. Degrees of freedom for $H_h^1, H_h^{\text{curl}}, H_h^{\text{div}}$ and L_h^2 for $k = 0$.

An important kind of finite element space is conforming finite element. If W is a given space of functions, then V_h is W-conforming if $V_h \subseteq W$. A natural question is how to get conforming global finite element space if we know how to construct local finit element spaces. The hint is given in the following theorem.

Theorem 120. *Assume that K_1 and K_2 are Lipchitz domain, $\Sigma = K_1 \cap K_2$, and function $u(x) \in \mathbb{R}^d$ ($d = 1, 2$ or 3) is given by:*

$$u(x) = \begin{cases} u_1(x), & x \in K_1 \\ u_2(x), & x \in K_1 \end{cases}$$

1. $u \in H^1(K_1 \cup K_2)$ iff u is continuous across Σ.
2. $\mathbf{u} \in H(curl, K_1 \cup K_2)$ iff $\mathbf{u_1} \times \boldsymbol{n} = \mathbf{u_2} \times \boldsymbol{n}$, where $\boldsymbol{n}$ is the normal vector of Σ pointing from K_1 to K_2.
3. $\mathbf{u} \in H(div, K_1 \cup K_2)$ iff $\mathbf{u_1} \cdot \boldsymbol{n} = \mathbf{u_2} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the normal vector of Σ pointing from K_1 to K_2.

Proof. Here we only give a proof of the second case. Proof of the first case can be found in the book of Ciarlet, and proof of the third case is similar to the second one. By integration by parts, we get:

$$\int_{K_1 \cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1 \cup K_2} \mathbf{u} \cdot \text{curl}\phi, \ \forall \phi \in C_0^\infty(K_1 \cup K_2)$$

By Stokes theorem:

$$\int_{K_1 \cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1} \mathbf{u_1} \cdot \text{curl}\phi + \int_{K_2} \mathbf{u_2} \cdot \text{curl}\phi + \int_\Sigma (\mathbf{u_1} \times \mathbf{n_1} + \mathbf{u_2} \times \mathbf{n_2}) dA$$

and $\mathbf{n_1} = \mathbf{n}$, $\mathbf{n_2} = -\mathbf{n}$, so:

$$\int_{K_1\cup K_2} curl\mathbf{u}\cdot\phi = \int_{K_1}\mathbf{u_1}\cdot \mathrm{curl}\phi + \int_{K_2}\mathbf{u_2}\cdot\mathrm{curl}\phi$$

Furthermore,

$$\|curl\mathbf{u}\|^2_{L^2(K_1\cup K_2)} = \|curl\mathbf{u_1}\|^2_{L^2(K_1)} + \|curl\mathbf{u_2}\|^2_{L^2(K_2)}$$

□

There are several important remarks about the above theorem. H^1-conforming elements are continuous for any dimension ($d = 1, 2, 3$). $H(curl)$-conforming elements have continuous tangential components across the boundary, while $H(div)$-conforming elements have continuous normal components across the boundary.

For the finite elements aforementioned, the nodal parameters, together with the shape functions, guarantees the conformity of the finite element space respectively.

18.3.2 A unified proof of the unisolvence

An important property related with finite element space is unisolvence.

Definition 14 (Unisolvence). *Given a mesh element $K \subseteq \mathbb{R}^d$ ($d = 1, 2, 3$), $\Sigma_K = \{l_1, l_2, \cdots, l_m\}$ (a set of linear functionals), and $\mathcal{P}_K$ is a set of linear polynomials, we say that $(K, \Sigma_K, \mathcal{P}_K)$ is unisolvent if specifying a value for each degree of freedoms uniquely determines an element in $\mathcal{P}_K$.*

Given Σ_K and $\mathcal{P}_K$, to verify unisolvence, we only need to show that the implication holds: if $v \in \mathcal{P}_K$ satisfies

$$l_1(v) = l_2(v) = \cdots = l_m(v) = 0$$

then $v = 0$.

Example. Assume that:

- $K = [a, a + h]$ for a given $a \in \mathbb{R}$.
- $\Sigma_K = \{u(a), u(a + h)\}$.
- $\mathcal{P}_K = \{\text{linear polynomials}\}$.

If $v \in \mathcal{P}_K$, $v(a) = v(a + h) = 0$, then for any $x \in (a, a + h)$:

$$v(x) = v(\alpha a + \beta(a + h)) = \alpha v(a) + \beta v(a + h) = 0$$

so $v = 0$.

Another important property of $(K, \Sigma_K, \mathcal{P}_K)$ is that: if $(K, \Sigma_K, \mathcal{P}_K)$ is unisolvent, then there exists a basis $\{\phi_i\}$ in $\mathcal{P}_K$ s.t.

$$l_j(\phi_i) = \delta_{ij}$$

Now we know the formal definition of finite element space on an element K in a given mesh, which is usually called local finite element space. Based on this, we can define global finite elemnt space.

Next we will prove the unisolvence property of the lowest order finite element spaces constructed above:

$$\begin{aligned} \mathcal{P}_0(K) &= \{a_0 + a_1x_1 + a_2x_2 + a_3x_3, a_i \in \mathbb{R}^1, 0 \le i \le 3\} \\ \mathcal{P}_1(K) &= \left\{\underset{\sim}{a_0} + \underset{\sim}{a_1} \times \underset{\sim}{x} \middle| \underset{\sim}{a_0} \in \mathbb{R}^3, \underset{\sim}{a_1} \in \mathbb{R}^3\right\} \\ \mathcal{P}_2(K) &= \left\{\underset{\sim}{a_0} + a\underset{\sim}{x} \middle| \underset{\sim}{a_0} \in \mathbb{R}^3, a \in \mathbb{R}^1\right\} \end{aligned} \tag{18.1}$$

- $k = 0$. $d_0 = \mathrm{grad} = \underset{\sim}{\nabla}$. The dofs are

$$\mathcal{N}_0(K) = \{v(a_i), 0 \le i \le 3\}.$$

The unisolvence means that

$$v(a_i) = 0, \forall i \;\Rightarrow\; v \equiv 0,$$

or

$$\text{There exists a unique } v \in \mathcal{P}_0(K), \text{ s.t. } v(a_i) = y_i, \ 0 \le i \le 3.$$

Proof. Take any edge $[a_i, a_j]$. We do the following integral

$$\int_{[a_i,a_j]} \underset{\sim}{\nabla} v ds = v(a_j) - v(a_i) = 0.$$

Note that $\underset{\sim}{\nabla} v$ is constant vector. We know that $\underset{\sim}{\nabla} v \cdot (a_i - a_j) = 0, \forall i, j$.
Since $\underset{\sim}{\nabla} v$ only has three degrees of freedoms and it is perpendicular to 6 edges, 3 of which are linearly independent. So $\underset{\sim}{\nabla} v = 0$ and furthermore, $v = 0$. □

- $k = 1$. d_1 = curl. The dofs are

$$\mathcal{N}_1(K) = \left\{ \int_{[a_i,a_j]} \underset{\sim}{v} \cdot \underset{\sim}{\tau}, \ i \neq j \right\}.$$

To prove the unisolvence, we need to show that

$$\int_{[a_i,a_j]} \underset{\sim}{v} \cdot \tau = 0, \forall i, j \ \Rightarrow \ \underset{\sim}{v} = 0.$$

Proof. Take any face of the tetrahedron $[a_i, a_j, a_k]$. By Stokes Theorem,

$$\iint_F \underset{\sim}{\nabla} \times \underset{\sim}{v} dN = \int_{\partial F} \underset{\sim}{v} \cdot \tau = 0.$$

Since $\underset{\sim}{\nabla} \times \underset{\sim}{v}$ is constant vector, we have

$$(\underset{\sim}{\nabla} \times \underset{\sim}{v}) \cdot n_F = 0, \ \forall F.$$

Here n_F is the normal vector of face F. We know that $\underset{\sim}{\nabla} \times \underset{\sim}{v}$ is perpendicular to the normal vectors of all the face, 3 of which are linearly independent. So $\underset{\sim}{v} = 0$. □

- $k = 2$. The dofs are

$$\mathcal{N}_2(K) = \left\{ \int_{[a_i,a_j,a_k]} \underset{\sim}{v} \cdot \underset{\sim}{n}, \ \forall [a_i, a_j, a_k] \right\}.$$

To prove the unisolvence, we need to show that

$$\int_{[a_i,a_j,a_k]} \underset{\sim}{v} \cdot \underset{\sim}{n} = 0, \forall i, j, k \ \Rightarrow \ \underset{\sim}{v} = 0.$$

Proof. The proof is straightforward by Stokes Theorem

$$\iiint_K \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \iint_{\partial K} \underset{\sim}{v} \cdot \underset{\sim}{n_F} = 0.$$

Then we know that $\underset{\sim}{v} = 0$. □

In any case, the nodal parameters and the shape functions match perfectly by the Stokes theorem.