6

$H(grad)$, $H(div)$ and $H(curl)$ Finite element spaces

6.1 Abstract definition of finite element

Ciarlet's finite element triplets will be described.

Definition 1. *The tripletet* $(K, \mathcal{P}, \mathcal{N})$ *is called a finite element where*

1. $K \subset R^n$ *is the element domain: a domain with picewise smooth boundary;*
2. $\mathcal{P}$ *is the space of shape functions: a finite dimensional space of functions on* K*;*
3. $\mathcal{N} = \{N_1, N_2, \ldots, N_m\}$ *is the nodal variables or degrees of freedom: a basis of* $\mathcal{P}'$.

Definition 2 (Unisolvent). *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a finite element. Given any* $\mu_i (1 \leq i \leq m)$*, there exists a unique function* $p \in \mathcal{P}$ *such that*

$$N_i(p) = \mu_i,\ 1 \leq i \leq m,$$

then $\mathcal{N}$ *is* $\mathcal{P}$*-unisolvent.*

It means that $\{N_i\}_{i=1}^m$ is linearly independent.

Lemma 30. $\mathcal{N}$ *is* $\mathcal{P}$*-unisolvent if and only if* $\{N_i\}_{i=1}^m$ *is a basis of* $\mathcal{P}'$.

Proof. " $\Rightarrow$ " is obvious since $\{N_i\}_{i=1}^m \in \mathcal{P}'$ is linearly independent.
$\Leftarrow$: For any $N \in \mathcal{P}'$, it can be represented as

$$N = \sum_{i=1}^{m} a_i N_i.$$

For any v such that $N_i(v) = 0, 1 \leq i \leq m$, we have

$$N(v) = \sum_{i=1}^{m} a_i N_i(v) = 0,\ \forall N \in \mathcal{P}'.$$

Thus, $v = 0$. □

Definition 3 (Nodal basis function). *Let* $(K, \mathcal{P}, \mathcal{N})$ *be a finite element.* $\{\phi_1, \phi_2, \ldots, \phi_m\}$ *is called the nodal basis for* $\mathcal{P}$ *if it is the basis for* $\mathcal{P}$ *that is duel to* $\mathcal{N}$*, namely* $N_i(\phi_j) = \delta_{ij}$.

Example 7. For the conforming linear element in $H^1(\Omega)$, let K be simplex, and $\mathcal{P}$ be the set of polynomials with degrees no larger than one, namely

$$\mathcal{P} = \mathcal{P}_1 = \{a_0 + a_1 x_1 + a_2 x_2 + a_3 x_3 : a_i \in \mathbb{R}\} = \{a + \tilde{a} \cdot \tilde{x}\}.$$

Let $\mathcal{N} = \{N_1, N_2, N_3, N_4\}$ where $N_i \in \mathcal{P}'$ and

$$N_i : v \to v(a_i)$$

be linear polynomials. Next, we show that the aforementioned triplet $(K, \mathcal{P}, \mathcal{N})$ is unisolvent.

Proof. If $v \in P_1$ such that $v(a_i) = 0,\ 1 \le i \le 4$. Let $e = [a_i, a_j]$ be an element edge of K. By stoke's theorem,

$$\int_e \mathrm{grad} v \cdot \tau = v(a_j) - v(a_i) = 0,$$

meaning,

$$\mathrm{grad} v \cdot (a_j - a_i) = 0, \forall 1 \le i < j \le 4$$

Thus, $\mathrm{grad} v = 0$ meaning v is constant. Since $v(a_i) = 0$, $v \equiv 0$. □

Definition 4 (Local interpolant). *The local interpolant associated with a finite element $(K, \mathcal{P}, \mathcal{N})$ togetherwise its nodal basis $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is defined as follows*

$$\mathcal{I}_K v = \sum_{i=1}^{m} N_i(v) \phi_i \tag{6.1}$$

for any function v for which all $N_i \in \mathcal{N}$ are well defined.

By definition, it is easy to see that the local interpolant is a linear operator that is invariant on $\mathcal{P}$.

Definition 5 (Global interpolant). *Let Ω be a domain that is partitioned by a collection of mutually disjoint open sets $\mathcal{T} = \{K_i\}$ such that $\bar{\Omega} = \cup \bar{K}_i$. Assume that each $K \in \mathcal{T}$ is associated with a finite element $(K, \mathcal{P}, \mathcal{N})$. For any function f in Ω for which each local interpolation $I_{K_i} f$ is well-defined, then the global interpolant associated with the partition $\mathcal{T}$ is simply piecing together the local interpolation as follows*

$$\mathcal{I}_{\mathcal{T}} f|_K = \mathcal{I}_K f, \quad K \in \mathcal{T}.$$

There are two types of the $H(\mathrm{curl})$ or $H(\mathrm{div})$ finite elements on simplicial grids: $\mathcal{P}_k^- \Lambda$ or $\mathcal{P}_k \Lambda$. In general, for the Hodge Laplacian, we have the four canonical pairs of mixed finite elements:

$$\mathcal{P}_{r+1}\Lambda^{k-1} \times \mathcal{P}_r \Lambda^k, \quad \mathcal{P}_{r+1}^-\Lambda^{k-1} \times \mathcal{P}_r \Lambda^k, \quad \mathcal{P}_r \Lambda^{k-1} \times \mathcal{P}_r^- \Lambda^k, \quad \mathcal{P}_r^- \Lambda^{k-1} \times \mathcal{P}_r^- \Lambda^k.$$

The choices can be viewed as the combination of the following two discrete complexes:

$$\begin{array}{ccccccc} \mathcal{P}_r^-\Lambda^0 & \xrightarrow{\mathrm{grad}} & \mathcal{P}_r^-\Lambda^1 & \xrightarrow{\mathrm{curl}} & \mathcal{P}_r^-\Lambda^2 & \xrightarrow{\mathrm{div}} & \mathcal{P}_r^-\Lambda^3 \\ \mathcal{P}_r\Lambda^0 & \xrightarrow{\mathrm{grad}} & \mathcal{P}_{r-1}\Lambda^1 & \xrightarrow{\mathrm{curl}} & \mathcal{P}_{r-2}\Lambda^2 & \xrightarrow{\mathrm{div}} & \mathcal{P}_{r-3}\Lambda^3. \end{array} \tag{6.2}$$

6.1.1 Examples of element domains

***d* = 1**

An element domain in 1D is just an interval.

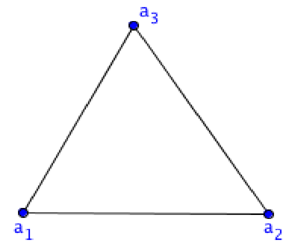

Fig. 6.1. A 2-simplex (triangle)

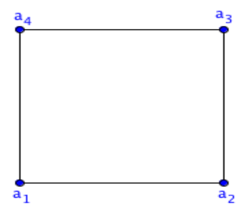

Fig. 6.2. A 2-simplex (rectangle)

$d = 2$

There are two commonly used element domains in 2D. One is 2D simplex or triangle, see Fig. 6.1, and another is rectangle, see Fig. 6.2.

$d = 3$

There are two commonly used element domains in 3D. One is 3D simplex or tetrahedron, see Fig. 6.3, and another is a hexahedron, see Fig. 6.4

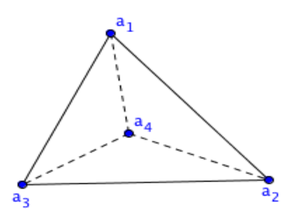

Fig. 6.3. A 3-simplex (tetrahedron)

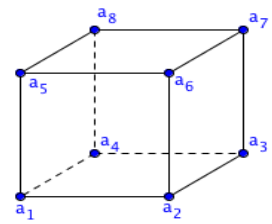

Fig. 6.4. A 3-simplex (hexahedron)

6.1.2 Space of shape functions

If the element domain is a simplex, the space of shape function is typically chosen to be complete or incomplete polynomial of certain degrees, see (**??**). If the element domain is a cube, the space of shape function is typically chosen to be complete or incomplete polynomial in the form of (**??**).

6.1.3 Barycentric coordinates

We note that an n-simplex τ in $\mathbb{R}^n$ is the convex hull τ of $n+1$ points $a_j = (a_{ij})_{i=1}^n \in \mathbb{R}^n$, which are called the vertices of the n-simplex, namely

$$\tau = \{x = \sum_{j=1}^{n+1} \lambda_j a_j; 0 \le \lambda_j \le 1, 1 \le j \le n+1, \sum_{j=1}^{n+1} \lambda_j = 1\}.$$

In the above, $\lambda_j = \lambda_j(x)$, $1 \le j \le n+1$, are called the barycentric coordinates of any point $x \in \tau$. It is easy to see that

$$\lambda_j(x) = |\tau_j(x)|/|\tau| \in P_1(\tau)$$

where $\tau_j(x)$ is the n-simplex with vertices x and all a_i except a_j, and $|\tau|$ is the measure of τ. One has

Lemma 31. *Given any linear function $v \in P_1(\tau)$. The v is uniquely determined by its values on the vertices of τ, namely $v(a_j)$, $1 \le j \le n+1$.*

Lemma 32. *The barycentric coordinates have the following properties*

1. $\lambda_i(a_j) = \delta_{ij}$.
2. $\sum_{j=1}^{n+1} \lambda_j(x) = 1$ *for all* $x \in \tau$.
3. $v(x) = \sum_{j=1}^{n+1} v_{a_j} \lambda_j(x)$ *for any* $v \in P_1(\tau)$.

6.2 Two general families of finite element spaces on tetrahedron

In this section, we discuss two general families of finite element spaces on tetrahedron. It is well-known that in $\mathbb{R}^d$, there are 2^{d-1} exact polynomial sequences. That is, there are two different types of polynomial shape functions for H_h^{curl} and H_h^{div}, and only one type of polynomial shape functions for H_h^1 and L_h^2.

6.2.1 Incomplete polynomial shape functions

In this section, we discuss about the incomplete polynomial shape functions. First of all, we give the degrees of freedom. Then we give the nodal basis functions in terms of barycentric coordinates in an arbitrary tetrahedron.

H_h^1: continuous piecewise polynomials of degree at most $k+1$ This is a standard continuous finite element space. On each tetrahedral element τ, the elements of $H_h^1$are arbitrary $\mathcal{P}_{k+1}$, where $\mathcal{P}_{k+1}$ denotes the space of polynomials of degree at most $k+1$. The degrees of freedom are given by (1) the values at the vertices, (2) the edge moments of order at most $k-1$ (more precisely the funtionals that associate to the finite element function its inner product in L^2 on the edge with each element of a basis for $\mathcal{P}_{k-1}$), (3) the face moments of order at most $k-2$, and (4) the tedrahedral moments of order at most $k-3$.
Nodal basis functions are given by:

$$\left\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4} :\ \alpha_i \geq 0,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 = k+1\right\}.$$

H_h^{curl}: the Nedelec edge discretization of $H(\mathrm{curl})$ of index k On each tetrahedon τ, the elements of $H_h^{\mathrm{curl}}$are functions of the form: $p(x)+r(x)$ with $p \in [\mathcal{P}_k]^3, r \in [\mathcal{P}_{k+1}]^3$ such that $r \cdot x = 0$. The degrees of freedom of $q \in Hhc$ are (1) the moments of $q \cdot s$ of order at most k on each edge, where s is the tangential direction of the edge, (2) the moments of $q \times n$ of order at most $k-1$ on each face, and (3) the moments of q at most $k-2$ on each tetrahedron.
Nodal basis functions are given by:

$$\left\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\phi_{jl} :\ j < l \in \{1,2,3,4\},\right.$$
$$\left.\alpha_1+\alpha_2+\alpha_3+\alpha_4 = k-1,\ \alpha_i \geq 0,\ \alpha_i = 0 \text{ if } i < j\right\},$$

where $\phi_{jl} = \lambda_j\nabla\lambda_l - \lambda_l\nabla\lambda_j$.

H_h^{div}: the Raviart-Thomas discretization of $H(\mathrm{div})$ of index k On each tetrahedon τ, the elements of $H_h^{\mathrm{div}}$are functions of the form: $p(x)+r(x)x$ with $p \in [\mathcal{P}_k]^3, r \in \mathcal{P}_k$. The degree of freedom of $u \in H(\mathrm{div})$ are (1) the moments of $u \cdot n$ of order at most k on each face, and, (2) the moments of u of degree $k-1$ on each tetrahedron.
Nodal basis functions are given by:

$$\left\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\phi_{jk} :\ j < l < m \in \{1,2,3,4\},\right.$$
$$\left.\alpha_1+\alpha_2+\alpha_3+\alpha_4 = k-1,\ \alpha_i \geq 0,\ \alpha_i = 0 \text{ if } i < j\right\},$$

where $\phi_{jlm} = \lambda_j\nabla\lambda_l \times \nabla\lambda_m - \lambda_l\nabla\lambda_j \times \nabla\lambda_m + \lambda_m\nabla\lambda_j \times \nabla\lambda_l$.

L_h^2: arbitrary piecewise polynomials of degree at most k On each tetrahedral element τ, the elements of $L_h^2$are arbitrary $\mathcal{P}_k$ and degrees of freedoms are the tetrahedral moments of order at most k.
Nodal basis functions are given by:

$$\left\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4} :\ \alpha_i \geq 0,\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 = k\right\}.$$

To make the above contents easier to understand, we take H_h^{curl} case as an example and give its nodal basis in the following table (for $1 \leq k \leq 2$). Assume that x_i ($i = 1,2,3,4$) are the vertices of a given tetrahedron τ, then we write $\tau = [x_i, x_j, x_l, x_m]$ ($i < j < l < m$). The edge with endpoints x_i and x_j ($i < j$) is denoted by $[x_i, x_j]$, and the face with vertices x_i, x_j, x_l ($i < j < l$) is denoted by $[x_i, x_j, x_l]$. Then the nodal basis are given by

k	Edge $[x_i, x_j]$	Face $[x_i, x_j, x_l]$	Tet $[x_i, x_j, x_l, x_m]$
1	ϕ_{ij}		
2	$\{\lambda_i, \lambda_j\}\phi_{ij}$	$\lambda_k\phi_{ij}, \lambda_j\phi_{ik}$	

6.2.2 Complete polynomial shape functions

As mentioned before, H_h^1 and L_h^2 are the same for both incomplete and complete polynomial shape functions. So we only consider H_h^{curl} and H_h^{div} in this section.

H_h^{curl}: the Nedelec 2nd kind discretization of $H(\text{curl})$ of index k On each tetrahedron τ, the elements of H_h^{curl} are the functions in $[\mathcal{P}_k]^3$. The degrees of freedom for $q \in H_h^{\text{curl}}$ are given by (1) the moments of $q \cdot s$ of order at most k on each edge, where s is the tangential direction of an edge, (2)$\int_f (q \times n) \cdot v$ for each face f, where $v \in \left\{[\mathcal{P}_{k-2}(f)]^2 \oplus \widetilde{\mathcal{P}}_{k-1}(f)x\right\}$. And $\widetilde{\mathcal{P}}_{k-1}(f)$ is the homogeneous polynomials space of order $k-1$ on a face f. (3) the moments of q of order at most $k-2$ on the tetrahedron.
Nodal basis functions are given by:

$$\left\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\nabla\lambda_j : \ j \in \{1,2,3,4\},\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 = k,\ \alpha_i \geq 0\right\}.$$

H_h^{div}: the Nedelec 2nd kind discretization of $H(\text{div})$ of index k On each tetrahedron τ, the elements of H_h^{curl} are the functions in $[\mathcal{P}_k]^3$. The degrees of freedom for $q \in H_h^{\text{div}}$ are given by (1) moments $q \cdot n$ of order at most k on each face, (2) $\int_\tau q \cdot v$ on the tetrahedron τ, where $v \in [\mathcal{P}_{k-2}]^3 \oplus \mathcal{S}_{k-1}$. Here, $\mathcal{S}_{k-1} = \{v \in \text{homogeneous polynomials of degree } k-1 : \ x \cdot v = 0\}$.
Nodal basis functions are given by:

$$\left\{\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\lambda_3^{\alpha_3}\lambda_4^{\alpha_4}\nabla\lambda_j \times \nabla\lambda_l : \ j < l \in \{1,2,3,4\},\ \alpha_1+\alpha_2+\alpha_3+\alpha_4 = k,\ \alpha_i \geq 0\right\}.$$

6.3 3D tetrahedron: lowest incomplete polynomials

In this subsection, we shall give a detailed discussions of all these elements for the lowest order case, namely $\mathcal{P}_1^-\Lambda$ and $\mathcal{P}_1\Lambda$. This simplest case is interesting in many ways.

The $H_h^1(\Omega) = \mathcal{P}_1^-\Lambda^0 = \mathcal{P}_1\Lambda^0$ is simply our familiar continuous piecewise linear finite element space and $L_h^2(\Omega) = \mathcal{P}_1^-\Lambda^3 = \mathcal{P}_0\Lambda^3$ is the piecewise constant finite element space. The descriptions of the spaces H_h^{curl} and H_h^{div}, however, require some works.

6.3.1 $\mathcal{P}_1\Lambda^0 \subset H(\text{grad};\Omega)$

6.3.2 Lowest order first type Nédélec element: $\mathcal{P}_1^-\Lambda^1 \subset H(\text{curl};\Omega)$

For $H_h^{\text{curl}} = \mathcal{P}_1^-\Lambda^1$, on each element K, the shape function space is 6-dimensional

$$\mathcal{P} := \mathcal{ND}_0 = \{\underset{\sim}{\alpha} + \underset{\sim}{\beta} \times \underset{\sim}{x} : \underset{\sim}{\alpha}, \underset{\sim}{\beta} \in \mathbb{R}^3\},$$

and the degrees of freedoms are tangential integral on each element edge

$$\mathcal{N} = \left\{\int_e \underset{\sim}{v} \cdot \underset{\sim}{t} : \text{for each edge } e \subset K\right\}.$$

Lemma 33. *Given any face $F \subset K$, if $\underset{\sim}{v} = \underset{\sim}{\alpha} + \underset{\sim}{\beta} \times \underset{\sim}{x} \in \mathcal{N}D_0$ is such that*

$$\int_e \underset{\sim}{v} \cdot \underset{\sim}{t} = 0 \text{ for each edge } e \subset F,$$

then

$$\underset{\sim}{\beta} \cdot \underset{\sim}{n} = 0. \tag{6.3}$$

and

$$\underset{\sim}{v} \times \underset{\sim}{n} = 0, \quad \text{on } F. \tag{6.4}$$

Proof. First (6.3) is valid because of that $\nabla \times \underset{\sim}{v} = 2\underset{\sim}{\beta}$ and the Stokes theorem:

$$\iint_F (\nabla \times \underset{\sim}{v}) \cdot \underset{\sim}{n} = \int_{\partial F} \underset{\sim}{v} \cdot \underset{\sim}{t} = 0.$$

Thus $\underset{\sim}{\beta} \cdot \underset{\sim}{n} = 0$.

Since

$$\underset{\sim}{n} \times (\underset{\sim}{\beta} \times \underset{\sim}{x}) = \underset{\sim}{\beta}(\underset{\sim}{n} \cdot \underset{\sim}{x}) - \underset{\sim}{x}(\underset{\sim}{n} \cdot \underset{\sim}{\beta}) = \underset{\sim}{\beta}(\underset{\sim}{n} \cdot \underset{\sim}{x}),$$

we have

$$\underset{\sim}{v} \times \underset{\sim}{n} = \underset{\sim}{\alpha} \times \underset{\sim}{n} - \underset{\sim}{\beta}(\underset{\sim}{n} \cdot \underset{\sim}{x}) = \text{const}.$$

Thus

$$(\underset{\sim}{v} \times \underset{\sim}{n}) \times \underset{\sim}{t}_e = (\underset{\sim}{t}_e \cdot \underset{\sim}{v})\underset{\sim}{n} - (\underset{\sim}{t}_e \cdot \underset{\sim}{n}) \cdot \underset{\sim}{v} = (\underset{\sim}{v} \cdot \underset{\sim}{t}_e)\underset{\sim}{n} = 0, \quad e \subset \partial F,$$

which leads to (6.4). □

Clearly, if all the local degrees of freedom vanish, (6.3) implies that $\underset{\sim}{\beta} = 0$, therefore $\underset{\sim}{\alpha} = 0$ by (6.4). Thus we prove the unisolvent.

Definition 6. *Given a tetrahedral finite element partition $\mathcal{T}_h$, let H_h^{curl} consist of function $\underset{\sim}{v}$ such that*

1. *$\underset{\sim}{v}|_K \in \mathcal{N}\mathcal{D}_0$ for any $K \in \mathcal{T}_h$;*
2. *$\underset{\sim}{v}$ has equal nodal variables (degrees of freedoms) at common faces (and edges).*

Using (6.4) and Lemma 29, we see the above defined finite element space satisfies

$$H_h^{\text{curl}} \subset H(\text{curl}; \Omega).$$

Basis functions

The basis function for tetrahedra mesh is as following: for each edge $e_{ij}, 1 \le i < j \le 4$, the basis functions are

$$\phi_{e_{ij}} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i, 1 \le i < j \le 4.$$

Proof. Let K be the tetrahedron with four vertices a_1, a_2, a_3, a_4. We first notice that the normal to the face f_{123} with vertices a_1, a_2, a_3 is parallel to $\nabla \lambda_4$. In fact, for any edge e of f_{123} with tangential vector t_e

$$\nabla \lambda_4 \cdot t_e = \frac{\partial \lambda_4}{\partial t_e} = 0.$$

Similarly, we have the normal to the face f_{234} is parallel to $\nabla\lambda_1$. Now we show that the basis function corresponding to the edge e_{14} is $\phi_{e_{14}} = \lambda_1\nabla\lambda_4 - \lambda_4\nabla\lambda_1$. In fact, since $\lambda_1 = 0$ on the face f_{234} and $\lambda_4 = 0$ on the face f_{123}, we have

$$\int_e \phi_{e_{14}} \cdot t_e ds = 0, \forall e \neq e_{14}.$$

It remain to prove

$$\int_{e_{14}} \phi_{e_{14}} \cdot t_{14} ds = 1.$$

Noting that $\lambda_1 + \lambda_4 = 1$, we have

$$\phi_{e_{14}} = (1-\lambda_4)\nabla\lambda_4 - \lambda_4\nabla\lambda_1 = \nabla\lambda_4 - \lambda_4\nabla(\lambda_1+\lambda_4) = \nabla\lambda_4 + \lambda_4\nabla(\lambda_2+\lambda_3) \text{ on } e_{14}.$$

Therefore

$$\int_{e_{14}} \phi_{e_{14}} \cdot t_{14} ds = \int_{e_{14}} \nabla\lambda_4 \cdot t_{14} ds = \int_{e_{14}} \frac{\partial\lambda_4}{\partial t_{14}} ds = 1.$$

This completes the proof. □

6.3.3 Lowest order Raviart-Thomas element: $\mathcal{P}_1^-\Lambda^2 \subset H(\mathrm{div};\Omega)$

For $H_h^{\mathrm{div}} = \mathcal{P}_1^0\Lambda^2$, on each element K, the shape function space is 4-dimensional

$$\mathcal{P} := \mathcal{RT}_0 = \{\underset{\sim}{\alpha} + \beta\underset{\sim}{x} : \underset{\sim}{\alpha} \in \mathbb{R}^3, \beta \in \mathbb{R}\},$$

and the degrees of freedoms are normal integral on each element face

$$\mathcal{N} = \left\{\iint_F \underset{\sim}{v} \cdot \underset{\sim}{n} : \text{ for each face } F \subset K\right\}.$$

To see the so-defined $(K, \mathcal{P}, \mathcal{N})$ is well-defined, consider any shape function $\underset{\sim}{v} = \underset{\sim}{\alpha} + \beta\underset{\sim}{x}$ that has all vanishing degrees of freedom

$$\iint_F \underset{\sim}{v} \cdot \underset{\sim}{n} = 0 \text{ for each face } F \subset K.$$

By divergence theorem

$$\iiint_K \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \iint_{\partial K} \underset{\sim}{v} \cdot \underset{\sim}{n} = 0.$$

As $\underset{\sim}{\nabla} \cdot \underset{\sim}{v} = 3\beta$, we have $\beta = 0$. From this, we can further deduce that $\underset{\sim}{\alpha} \cdot \underset{\sim}{n} = 0$ and hence $\underset{\sim}{v} = 0$. Thus we proved that only zero function can have vanishing degrees of freedom and hence the finite element $(K, \mathcal{P}, \mathcal{N})$ is well-defined.

Definition 7. *Given a tetrahedral finite element partition $\mathcal{T}_h$, let H_h^{div} consist of function $\underset{\sim}{v}$ such that*

1. *$\underset{\sim}{v}|_K \in \mathcal{RT}_0$ for any $K \in \mathcal{T}_h$;*
2. *v has equal nodal variables (degrees of freedoms) at common faces (and edges).*

Using the fact that each function $\underset{\sim}{v} \in \mathcal{RT}_0$ has constant normal component on each face, we see the above defined finite element space satisfies

$$H_h^{\mathrm{div}} \subset H(\mathrm{div};\Omega).$$

Basis functions

The basis function for tetrahedra mesh is as following: for each face $f_{ijk}, 1 \le i < j < k \le 4$, the basis functions are

$$\phi_{f_{ijk}} = \lambda_i \nabla\lambda_j \times \nabla\lambda_k + \lambda_k \nabla\lambda_i \times \nabla\lambda_j + \lambda_j \nabla\lambda_k \times \nabla\lambda_i, 1 \le i < j < k \le 4.$$

Proof. Let K be the tetrahedron with four vertices a_1, a_2, a_3, a_4. We first notice that the normal to the face f_{124} with vertices a_1, a_2, a_3 is parallel to $\nabla\lambda_3$. In fact, for any edge e of f_{124} with tangential vector t_e

$$\nabla\lambda_3 \cdot t_e = \frac{\partial\lambda_3}{\partial t_e} = 0.$$

Similarly, we have the normal to the face f_{234} is parallel to $\nabla\lambda_1$, the normal to the face f_{123} is parallel to $\nabla\lambda_4$, and the normal to the face f_{134} is parallel to $\nabla\lambda_2$. Now we show that the basis function corresponding to the edge f_{123} is $\phi_{f_{123}} = \lambda_1\nabla\lambda_2 \times \nabla\lambda_3 + \lambda_3\nabla\lambda_1 \times \nabla\lambda_2 + \lambda_2\nabla\lambda_3 \times \nabla\lambda_1$. In fact, since $\lambda_3 = 0$ on the face f_{124}, we have

$$\begin{aligned}
\int_{f_{124}} \phi_{f_{123}} \cdot n_{f_{124}} dA &= \int_{f_{124}} (\lambda_1\nabla\lambda_2 \times \nabla\lambda_3 + \lambda_3\nabla\lambda_1 \times \nabla\lambda_2 + \lambda_2\nabla\lambda_3 \times \nabla\lambda_1) \cdot n_{f_{124}} dA \\
&= \int_{f_{124}} (\lambda_1\nabla\lambda_2 \times \nabla\lambda_3 + \lambda_2\nabla\lambda_3 \times \nabla\lambda_1) \cdot n_{f_{124}} dA \\
&= \int_{f_{124}} (\lambda_1\nabla\lambda_2 \times \nabla\lambda_3 + \lambda_2\nabla\lambda_3 \times \nabla\lambda_1) \cdot n_{f_{124}} dA \\
&= \int_{f_{124}} \lambda_1 \, (\nabla\lambda_3 \times n_{f_{124}}) \cdot \nabla\lambda_2 + \lambda_2 \, (n_{f_{124}} \times \nabla\lambda_3) \cdot \nabla\lambda_1 dA \\
&= \int_{f_{124}} \lambda_1 \, 0 \cdot \nabla\lambda_2 + \lambda_2 \, 0 \cdot \nabla\lambda_1 dA = 0
\end{aligned} \tag{6.5}$$

Similarly, we have

$$\int_f \phi_{f_{123}} \cdot n_f ds = 0, \forall f \ne f_{123}.$$

It remain to prove

$$\int_{f_{123}} \phi_{f_{123}} \cdot n_{f_{123}} ds \ne 0.$$

Noting that $\lambda_1 + \lambda_2 + \lambda_3 = 1$ on f_{123}, we have

$$\begin{aligned}
\phi_{f_{123}} &= \lambda_1\nabla\lambda_2 \times \nabla\lambda_3 + (1 - \lambda_1 - \lambda_2)\nabla\lambda_1 \times \nabla\lambda_2 + \lambda_2\nabla\lambda_3 \times \nabla\lambda_1 = \nabla\lambda_1 \times \nabla\lambda_2 \\
&= \nabla\lambda_1 \times \nabla\lambda_2 + \lambda_1\nabla\lambda_2 \times \nabla\lambda_4 + \lambda_2\nabla\lambda_1 \times \nabla\lambda_4 \\
&= \nabla \times (\lambda_1\nabla\lambda_2) + \lambda_1\nabla\lambda_2 \times \nabla\lambda_4 + \lambda_2\nabla\lambda_1 \times \nabla\lambda_4.
\end{aligned} \tag{6.6}$$

Therefore, noting that $\nabla\lambda_4$ is parallel to $n_{f_{123}}$ and by Stoke's theorem, we have

$$\begin{aligned}
\int_{f_{123}} \phi_{f_{123}} \cdot n_{f_{123}} dA &= \int_{f_{123}} \nabla \times (\lambda_1\nabla\lambda_2) \cdot n_{f_{123}} dA = \int_{\partial f_{123}} \lambda_1\nabla\lambda_2 \cdot t ds = \int_{e_{12}} \lambda_1\nabla\lambda_2 \cdot t ds \\
&= \int_{e_{12}} \lambda_1 \frac{\partial\lambda_2}{\partial t} ds = \frac{1}{|e_{12}|} \int_{e_{12}} \lambda_1 ds = \frac{1}{2}.
\end{aligned} \tag{6.7}$$

This completes the proof. □

6.4 3D tetrahedron: lowest complete polynomials

6.4.1 Quadratic H^1 element: $\mathcal{P}_2\Lambda^0 \subset H^1(\Omega)$

On each element K, the shape function space is P_2, and the degrees of freedoms is composed of two sets. For a function u on K such that $u \in (H^{3/2+s}(K))^3$, $s > 0$, we define

$$M_v(u) = \Big\{u(a_i), 1 \le i \le 4\Big\}, \tag{6.8}$$

$$M_e(u) = \Big\{\int_e u ds \text{ for each edge } e \text{ of } K\Big\}, \tag{6.9}$$

Unisolvent

Lemma 34. *If $u \in P_2$ and all the degrees of freedom in* (6.8) *for u associated with a given face f vanish, then $u = 0$ on f.*

Proof. Since the vertex degrees of freedom for u on the given face f vanish, for each $e \in \partial f$ we may write

$$\int_e \frac{\partial u}{\partial s} q ds = -\int_e u \frac{\partial q}{\partial s} ds = 0 \tag{6.10}$$

for any $q \in P_1(e)$. Chooseing $q = \frac{\partial u}{\partial s}$ shows that $\frac{\partial u}{\partial s} = 0$ along this edge and hence $u = 0$ along each edge. Since $u = 0$ on $e_1, e_2 \subset \partial f$, $u = c\lambda_1\lambda_2$, where λ_1, λ_2 are the area barycentric coordinate functions for f. Since $u = 0$ on $e_3 \subset f$, hence $c = 0$, and we have proved that $u = 0$ on f. □

Lemma 35. *If $u \in P_2$ and all the degrees of freedom in* (6.8) *and* (6.9) *vanish, then $u = 0$.*

Proof. By Lemma 34, $u = 0$ on each face f of K. Namely $u = 0$ on $f_1, f_2 \subset K$. Thus $u = c\lambda_1\lambda_2$ for some constant c, where λ_1, λ_2 are the volume barycentric coordinate functions for K. Since $u = 0$ on f_3, hence $c = 0$, and we have proved that $u = 0$. □

Basis functions

The basis functions for tetrahedra mesh are as following: for each vertex $a_i, 1 \le i \le 4$, the basis function is

$$\phi_i = \lambda_i(\lambda_i - \frac{2}{3}), 1 \le i \le 4.$$

For each edge e_{ij}, the basis function is

$$\phi_{e_{ij}} = \lambda_i\lambda_j, 1 \le i < j \le 4.$$

6.4.2 Lowest order second type Nédélec element: $\mathcal{P}_1\Lambda^1 \subset H(\mathbf{curl}; \Omega)$

On each element K, the shape function space is $\mathcal{P} := \mathcal{N}C_1 = (P_1)^3$. The degrees of freedom are defined as follows: For a function $\underset{\sim}{v}$ on K such that $\underset{\sim}{v} \in (H^{1/2+s}(K))^3, \delta > 0$ and $\nabla \times \underset{\sim}{v} \in (L^q(K))^3, q > 2$, we define

$$\mathcal{N} = \Big\{\int_e (\underset{\sim}{v} \cdot \underset{\sim}{t}) q\, ds \text{ for all } q \in P_1(e) \text{ for each edge } e \text{ of } K\Big\}. \tag{6.11}$$

It is not difficult to check that the total number of degrees of freedom in $\mathcal{N}$ is 12, which equals to the dimension of $\mathcal{N}C_1$. To see the finite element triple $(K, \mathcal{P}, \mathcal{N})$ is well-defined, we consider function $\underset{\sim}{v} \in \mathcal{P}$

such that all of its degrees of freedom vanish Since $\underset{\sim}{v} \cdot \underset{\sim}{t}|_e \in P_1(e)$, we know that $\underset{\sim}{v} \cdot \underset{\sim}{t} = 0$ on e. By the Stokes theorem:

$$\iint_F (\nabla \times \underset{\sim}{v}) \cdot n = \int_{\partial F} \underset{\sim}{v} \times \underset{\sim}{t} = 0,$$

which gives $\nabla \times \tilde{v} = 0$ since $\nabla \times \underset{\sim}{v}$ is constant. By the exactness, there exists $p \in P_2(T)$ such that $\underset{\sim}{v} = \nabla p$. Thus, p is constant along edge e since $\nabla p \cdot \underset{\sim}{t} = \underset{\sim}{v} \cdot \underset{\sim}{t} = 0$. Hence, p is constant on T which implies that $\underset{\sim}{v} = 0$.

From the proof, we immediately see that

$$\mathcal{P}_1 \Lambda^1 \subset H(\mathrm{curl}; \Omega).$$

Basis functions

The basis functions for tetrahedra mesh are as following: for each edge e_{ij}, the basis functions are

$$\phi_{1,e_{ij}} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i.$$

$$\phi_{2,e_{ij}} = \lambda_i \nabla \lambda_j + \lambda_j \nabla \lambda_i.$$

6.4.3 Lowest order Brezzi-Douglas-Marini element: $\mathcal{P}_1 \Lambda^2 \subset H(\mathrm{div}; \Omega)$

On each element K, the shape function space is $\mathcal{P} := \mathcal{BDM}_1 = (P_1)^3$, and the degrees of freedoms is defined as follows: For a function $\underset{\sim}{v}$ on K such that $\underset{\sim}{v} \in (H^{1/2+s}(K))^3, \delta > 0$, we define

$$\mathcal{N} = \left\{ \iint_F (\underset{\sim}{v} \cdot \underset{\sim}{n}) q dA \text{ for all } q \in P_1(F) \text{ for each face } F \text{ of } K \right\}. \tag{6.12}$$

To show that the finite element triple $(K, \mathcal{P}, \mathcal{N})$ is well-defined, we consider function $\underset{\sim}{v} \in \mathcal{P}$ such that all of its degrees of freedom vanish, i.e.,

$$\iint_F (\underset{\sim}{v} \cdot \underset{\sim}{n}) q dA = 0, \forall q \in P_1(F) \text{ for each edge } F \subset K.$$

Since $\underset{\sim}{v} \cdot \underset{\sim}{n}|_F \in P_1(F)$, we know that $\underset{\sim}{v} \cdot \underset{\sim}{n} = 0$ on ∂K. Note that the bubble function has the form,

$$\underset{\sim}{v} = \sum_{ij} \lambda_i \lambda_j \underset{\sim}{t}_{ij} p_{ij},$$

which means that $\underset{\sim}{v} = 0$. So this finite element triple is uni-solvent. From the proof, we see that

$$\mathcal{P}_1 \Lambda^2 \subset H(\mathrm{div}; \Omega).$$

Basis functions

The basis functions for tetrahedra mesh are as following: for each face f_{ijk}, the basis functions are

$$\phi_{1,f_{ijk}} = \lambda_i \nabla \lambda_j \times \nabla \lambda_k + \lambda_k \nabla \lambda_i \times \nabla \lambda_j + \lambda_j \nabla \lambda_k \times \nabla \lambda_i.$$

$$\phi_{2,f_{ijk}} = \lambda_i \nabla \lambda_j \times \nabla \lambda_k - \lambda_k \nabla \lambda_i \times \nabla \lambda_j + \lambda_j \nabla \lambda_k \times \nabla \lambda_i.$$

$$\phi_{3,f_{ijk}} = \lambda_i \nabla \lambda_j \times \nabla \lambda_k - \lambda_k \nabla \lambda_i \times \nabla \lambda_j - \lambda_j \nabla \lambda_k \times \nabla \lambda_i.$$

6.5 2D triangle: lowest incomplete polynomials

6.5.1 Examples of local finite elements on a 2D triangle with incomplete polynomials

In 2-d case, we have the similar finite element spaces. For $k = 0$, the H_h^1 and H_h^{curl} is simply our familiar continuous piecewise linear finite element spaces and the L_h^2 is the piecewise constant finite element space. While the spaces H_h^{rot} and H_h^{div} require some work.

The space H_h^{rot}

For H_h^{rot}, on each triangle element K, the shape function space is 3-dimensional

$$\mathcal{P} = \left\{\alpha + \beta\begin{pmatrix}-y\\ x\end{pmatrix} : \ \alpha \in \mathbb{R}^2,\ \beta \in \mathbb{R}\right\},$$

and the degrees of freedom are the tangential integral on each edge

$$\mathcal{N} = \left\{\int_e v \cdot t dS \ : \ \text{for each edge } e \subset K\right\}.$$

To show that the finite element triple $(K, \mathcal{P}, \mathcal{N})$ is well-defined, we consider function $v \in \mathcal{P}$ such that all of its degrees of freedom vanish

$$\int_e v \cdot t dS = 0, \ \text{for each edge } e \subset K.$$

By Stoke's theorem,

$$\iint_K \text{rot } v dx = \int_{\partial K} v \cdot t dS = 0.$$

Since rot $v = 2\beta$, we know that $\beta = 0$. Furthermore, we can conclude that $\alpha = 0$. Therefore, $v = 0$. So this finite element triple is uni-solvent. It is easy to show that

$$H_h^{\text{rot}} \subset H(\text{rot}; \Omega).$$

Basis functions

Let l_1, l_2 and l_3 be the length of each edge e_1, e_2 and e_3 respectively. Define $\mathbf{W}_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$ for $i, j = 1, 2, 3$ and $i \neq j$. Then we have

1. div $\mathbf{W}_{ij}$ = div $(\lambda_i \nabla \lambda_j)$ − div $(\lambda_j \nabla \lambda_i) = 0$.
2. curl$\mathbf{W}_{ij}$ = curl$(\lambda_i \nabla \lambda_j)$ − curl$(\lambda_j \nabla \lambda_i) = 2\nabla \lambda_j \times \nabla \lambda_i$.
3. $\mathbf{e}_{ij} \cdot \nabla \lambda_i = -\frac{1}{l_{ij}}$ and $\mathbf{e}_{ij} \cdot \nabla \lambda_j = \frac{1}{l_{ij}}$. Then $\mathbf{e}_{ij} \cdot \mathbf{W}_{ij} = \frac{\lambda_i + \lambda_j}{l_{ij}} = \frac{1}{l_{ij}}$ and $\mathbf{W}_{ij}$ has no tangential component along the other two edges.

Thus the basis function for each edge e_{ij} is $\mathbf{W}_{ij} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i$.

The space H_h^{div}

For $k = 0$, the shape function space for $H_h(\text{div})$ on each triangle element K is 3-dimensional

$$\mathcal{P} = \left\{\alpha + \beta\begin{pmatrix}x\\y\end{pmatrix} : \ \alpha \in \mathbb{R}^2, \beta \in \mathbb{R}\right\},$$

and the degrees of freedom are normal integral on each element edge

$$\mathcal{N} = \left\{\int_e v \cdot n dS \ : \ \text{for each face } e \subset K\right\}.$$

To see the finite element triple $(K, \mathcal{P}, \mathcal{N})$ is well-defined, we consider function $v \in \mathcal{P}$ such that all of its degrees of freedom vanish

$$\int_e v \cdot n dS = 0, \ \text{for each edge } e \subset K.$$

By divergence theorem in 2D,

$$\iint_K \nabla \cdot v dx = \int_{\partial K} v \cdot n dS = 0.$$

Since $\nabla \cdot v = 2\beta$, we know that $\beta = 0$. And we can deduce that $\alpha = 0$. Therefore, $v = 0$. So this finite element triple is uni-solvent. It is easy to show that

$$H_h^{\text{div}} \subset H(\text{div}; \Omega)$$

Basis functions

The basis function for tetrahedra mesh is as following: for each edge e_{ij}, the basis function is

$$\phi_{e_{ij}} = \lambda_i \text{curl}\lambda_j - \lambda_j \text{curl}\lambda_i.$$

Proof. The average of $\phi_{e_{ij}} \cdot n$ is: on the edge e_{ij}, $\lambda_i + \lambda_j = 1$, so we replace this λ_j by $1 - \lambda_i$, then have

$$\begin{aligned}\frac{1}{|e_{ij}|}\int_{e_{ij}} \phi_{e_{ij}} \cdot \boldsymbol{n} ds &= \frac{1}{|e_{ij}|}\int_{e_{ij}} (\lambda_i(\text{curl}\lambda_j + \text{curl}\lambda_i) - \text{curl}\lambda_i) \cdot \boldsymbol{n}_k ds \\ &= \frac{1}{|e_{ij}|}\int_{e_{ij}} (-\lambda_i \text{curl}\lambda_k - \text{curl}\lambda_i) \cdot \boldsymbol{n}_k ds.\end{aligned}$$

Since $\text{curl}\lambda_k \cdot \boldsymbol{n}_k = \frac{\partial \lambda_k}{\partial t_k} = 0$, we can remove the first term.

$$\text{curl}\lambda_i \cdot \boldsymbol{n}_k = (R\nabla\lambda_i) \cdot \boldsymbol{n}_k = \nabla\lambda_i \cdot t_k = \frac{\partial \lambda_i}{\partial t_k}$$

$$\frac{1}{|e_{ij}|}\int_{e_{ij}} \phi_{e_{ij}} \cdot \boldsymbol{n} ds = -\frac{1}{|e_{ij}|}\int_{e_{ij}} \frac{\partial \lambda_i}{\partial t_k} ds = 1.$$

For edge e_{ik}, since λ_j equals zero on edge e,

$$\int_{e_{ik}} \phi_{e_{ij}} \cdot \boldsymbol{n} ds = \int_{e_{ik}} (\lambda_i \text{curl}\lambda_j - \lambda_j \text{curl}\lambda_i) \cdot \boldsymbol{n}_j ds = 0.$$

□

Commutative diagram

We first verify the commutative property in this diagram.

Lemma 36. *For any $v \in C^\infty(\Omega)$,*

$$\mathrm{div}\Pi_h^{\mathrm{div}} v = \Pi_h^0 \mathrm{div} v.$$

Proof. Since both $\mathrm{div}\Pi_h^{\mathrm{div}} v$ and $\Pi_h^0 \mathrm{div} v$ are piecewise constant, we only need to verify that they have the same degrees of freedom. By the definition of Π_h^{div} and Π_h^0, stokes theorem,

$$\begin{aligned}\int_K \mathrm{div}\Pi_h^{\mathrm{div}} v - \Pi_h^0 \mathrm{div} v dx &= \int_K \mathrm{div}\Pi_h^{\mathrm{div}} v - \mathrm{div} v dx \\ &= \int_{\partial K} \Pi_h^{\mathrm{div}} v \cdot n - v \cdot n ds = 0.\end{aligned}$$

Since both $\mathrm{div}\Pi_h^{\mathrm{div}} v$ and $\Pi_h^0 \mathrm{div} v$ are piecewise constant, and they have the same d.o.f, so

$$\mathrm{div}\Pi_h^{\mathrm{div}} v = \Pi_h^0 \mathrm{div} v.$$

□

Lemma 37. *For any $v \in C^\infty(\Omega)$,*

$$\mathrm{curl}\Pi_h^{\mathrm{curl}} v = \Pi_h^{\mathrm{div}} \mathrm{curl} v.$$

Proof. Since $\mathrm{curl}\Pi_h^{\mathrm{curl}} u$ is piecewise constant and belongs to H_h^{div}, we only need to prove that $\mathrm{curl}\Pi_h^{\mathrm{curl}} u$ has the same degrees of freedom with $\mathrm{div}(\mathrm{curl} u)$,

$$\begin{aligned}\int_{e_{ij}} (\mathrm{curl}\Pi_h^{\mathrm{curl}} u) \cdot n ds &= -\int_{e_{ij}} \frac{\partial}{\partial t} \Pi_h^{\mathrm{curl}} u ds \\ &= \Pi_h^{\mathrm{curl}} u(a_j) - \Pi_h^{\mathrm{curl}} u(a_i) \\ &= u(a_j) - u(a_i) \\ &= -\int_{e_{ij}} \frac{\partial}{\partial t} u ds \\ &= \int_{e_{ij}} (\mathrm{curl} u) \cdot n ds \\ &= \int_{e_{ij}} \Pi_h^{\mathrm{div}} (\mathrm{curl} u) \cdot n ds\end{aligned}$$

so

$$\mathrm{curl}\Pi_h^{\mathrm{curl}} v = \Pi_h^{\mathrm{div}} \mathrm{curl} v.$$

□

6.6 2D triangles: complete polynomials

On 2d triangle with complete polynomials, we have the similar finite element spaces. For $k = 0$, the H_h^1 and H_h^{curl} are still simply our familiar continuous piecewise linear finite element spaces and the L_h^2 is the piecewise constant finite element space. While the spaces H_h^{rot} and H_h^{div} also require some work.

The space H_h^{rot}

For H_h^{rot}, on each triangle element K with complete polynomial, the shape function space is 6-dimensional

$$\mathcal{P} = (P_1(K))^2$$

and the degrees of freedom are the tangential integral on each edge

$$\mathcal{N} = \left\{ \int_e v \cdot tqdS \ : \forall q \in P_1(e) \text{ for each edge } e \subset K \right\}.$$

To show that the finite element triple $(K, \mathcal{P}, \mathcal{N})$ is well-defined, we consider function $v \in \mathcal{P}$ such that all of its degrees of freedom vanish

$$\int_e v \cdot tqdS \ = 0, \forall q \in P_1(e) \text{ for each edge } e \subset K.$$

Since $v \cdot t|_e \in P_1(e)$, we know that $v \cdot t = 0$ on ∂K. Therefore, $v = 0$. So this finite element triple is uni-solvent. It is easy to show that

$$H_h^{\text{rot}} \subset H(\text{rot}; \Omega).$$

Basis functions

The basis functions for triangle mesh are as following: for each edge e_{ij}, the basis functions are

$$\phi_{1,e_{ij}} = \lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i.$$

$$\phi_{2,e_{ij}} = \lambda_i \nabla \lambda_j + \lambda_j \nabla \lambda_i.$$

The space H_h^{div}

For $k = 0$, the shape function space for $H_h(\text{div})$ on each triangle element K with complete polynomial is 6-dimensional

$$\mathcal{P} = (P_1(K))^2$$

and the degrees of freedom are normal integral on each element edge

$$\mathcal{N} = \left\{ \int_e v \cdot nqdS \ : \forall q \in P_1(e) \text{ for each edge } e \subset K \right\}.$$

To see the finite element triple $(K, \mathcal{P}, \mathcal{N})$ is well-defined, we consider function $v \in \mathcal{P}$ such that all of its degrees of freedom vanish Since $v \cdot t|_e \in P_1(e)$, we know that $v \cdot t = 0$ on ∂K. Therefore, $v = 0$. So this finite element triple is uni-solvent. It is easy to show that

$$H_h^{\text{div}} \subset H(\text{div}; \Omega)$$

The basis functions for triangle mesh are as following: for each edge e_{ij}, the basis functions are

$$\phi_{1,e_{ij}} = \lambda_i \text{curl} \lambda_j - \lambda_j \text{curl} \lambda_i.$$

$$\phi_{2,e_{ij}} = \lambda_i \text{curl} \lambda_j + \lambda_j \text{curl} \lambda_i.$$

6.7 3D tetrahedrons: finite elements of arbitrary orders

For clarity, details are only given for the descriptions of $H(\text{div}; \Omega)$ and $H(\text{curl}; \Omega)$ in three dimensions for simplicial elements. Results for two dimensions and other type of elements (such as tensor product elements) will only be listed without proofs.

Let us assume that Ω is a polyhedron in R^3 and it is partitioned by a quasi-uniform grid T_h. Fix an integer k, we introduce below the following finite element spaces for H^1, $H(\text{div})$, $H(\text{curl})$ and L^2 respectively. To define these spaces, we specify the corresponding polynomial spaces used on each element and the corresponding sets of degrees of freedom. The degrees of freedom for these spaces, denoted by $H_h^1, H_h^{\text{curl}}, H_h^{\text{div}}$ and L_h^2, for the lowest order case $k = 0$ are shown in Fig. 19.3.

6.7.1 Incomplete polynomials

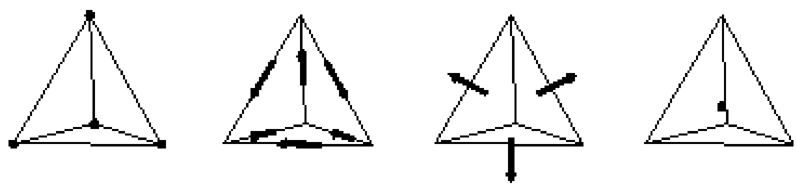

Fig. 6.5. Degrees of freedom for $H_h^1, H_h^{\text{curl}}, H_h^{\text{div}}$ and L_h^2 for $k = 0$.

H_h^1: continuous piecewise polynomials of degree at most $k + 1$ This is a standard continuous finite element space. On each tetrahedral element τ, the elements of H_h^1 are arbitrary $\mathcal{P}_{k+1}$, where $\mathcal{P}_{k+1}$ denotes the space of polynomials of degree at most $k + 1$. The degrees of freedom are given by (1) the values at the vertices, (2) the edge moments of order at most $k - 1$ (more precisely the funtionals that associate to the finite element function its inner product in L^2 on the edge with each element of a basis for $\mathcal{P}_{k-1}$), (3) the face moments of order at most $k - 2$, and (4) the tedrahedral moments of order at most $k - 3$.
Nodal basis functions are given by: [c1]

[c1] To be added here, from Arnold

H_h^{curl}: the Nedelec edge discretization of $H(\text{curl})$ of index k on each tetrahedon τ, the elements of H_h^{curl} are functions of the form: $p(x) + r(x)$ with $p \in [\mathcal{P}_k]^3$, $r \in [\mathcal{P}_{k+1}]^3$ such that $r \cdot x = 0$. The degrees of freedom of $q \in H_h^{\text{curl}}$ are (1) the moments of $q \cdot s$ of order at most k on each edge, where s is the tangential direction of the edge, (2) the moments of $q \times n$ of order at most $k - 1$ on each face, and (3) the moments of q at most $k - 2$ on each tetrahedron.
Nodal basis functions are given by: [c2]

[c2] To be added here, from Arnold

H_h^{div}: the Raviart-Thomas discretization of $H(\text{div})$ of index k on each tetrahedon τ, the elements of H_h^{div} are functions of the form: $p(x) + r(x)x$ with $p \in [\mathcal{P}_k]^3$, $r \in \mathcal{P}_k$. The degree of freedom of $u \in H(\text{div})$ are (1) the moments of $u \cdot n$ of order at most k on each face, and, (2) the moments of u of degree $k - 1$ on each tetrahedron.
Nodal basis functions are given by: [c3]

[c3] To be added here, from Arnold

L_h^2: arbitrary piecewise polynomials of degree at most k on each tetrahedral element τ, the elements of L_h^2 are arbitrary $\mathcal{P}_k$ and degrees of freedoms are the tetrahedral moments of order at most k.
Nodal basis functions are given by: [c4]

[c4] To be added here, from Arnold

6.7.2 Examples of local finite elements on a 3D tetrahedron with complete polynomials

The space $H^1(K)$

Definition

For $H^1(K)$, on each element K, the shape function space is P_k, and the degrees of freedoms is composed of four sets. For a function u on K such that $u \in (H^{3/2+s}(K))^3$, $s > 0$, we define

$$M_v(u) = \Big\{u(a_i), 1 \le i \le 4\Big\}, \tag{6.13}$$

$$M_e(u) = \Big\{\int_e uqds \text{ for all } q \in P_{k-2}(e) \text{ for each edge } e \text{ of } K\Big\}, \tag{6.14}$$

$$M_f(u) = \Big\{\int_f uqdA \text{ for all } q \in P_{k-3}(f) \text{ for each face } f \text{ of } K\Big\}, \tag{6.15}$$

$$M_K(u) = \Big\{\int_K uqdV \text{ for all } q \in P_{k-4}\Big\}. \tag{6.16}$$

Unisolvent

Lemma 38. *The degrees of freedom* (6.13) *and* (6.16) *for u on K vanish if and only if the degrees of freedom for $\hat{u}$ on $\hat{K}$ vanish.*

Lemma 39. *If $u \in P_k$ and all the degrees of freedom in* (6.13) *for u associated with a given face f vanish, then $u = 0$ on f.*

Proof. Since the vertex degrees of freedom for u on the given face f vanish, for each $e \in \partial f$ we may write

$$\int_e \frac{\partial u}{\partial s} qds = -\int_e u\frac{\partial q}{\partial s} ds = 0 \tag{6.17}$$

for any $q \in P_{k-1}(e)$. Chooseing $q = \frac{\partial u}{\partial s}$ shows that $\frac{\partial u}{\partial s} = 0$ along this edge and hence $u = 0$ along each edge. Since $u = 0$ on $\partial f, u = \lambda_1\lambda_2\lambda_3 r$, where $\lambda_1, \lambda_2, \lambda_3$ are the area harycentric coordinate functions for f. Then using the facts that $r \in P_{k-3}(f)$ and that the face degrees of freedom vanish, we have

$$0 = \int_f uqdA = \int_f \lambda_1\lambda_2\lambda_3 r^2 dA. \tag{6.18}$$

Hence $r = 0$, and we have proved that $u = 0$ on f. □

Lemma 40. *If $u \in P_k$ and all the degrees of freedom in* (6.20) *and* (6.21) *vanish, then $u = 0$.*

Proof. By Lemma 39, $u = 0$ on each face f of K. Namely $u = 0$ on ∂K. Thus $u = \lambda_1\lambda_2\lambda_3\lambda_4 r$ for some $r \in P_{k-4}$, where $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are the volume barycentric coordinate functions for K. Using the volume degrees of freedom

$$0 = \int_K urdV = \int_K \lambda_1\lambda_2\lambda_3\lambda_4 r^2 dV. \tag{6.19}$$

Hence $r = 0$, and we have proved that $u = 0$. □

The space $H_h^{\text{div}}(K)$

Definition

For H_h^{div}, on each element K, the shape function space is $(P_k)^3$, and the degrees of freedoms is composed of two sets. For a function u on K such that $u \in (H^{1/2+s}(K))^3, \delta > 0$, we define

$$M_f(u) = \left\{ \int_f (u \cdot n) q dA \text{ for all } q \in P_k(f) \text{ for each face } f \text{ of } K \right\}, \tag{6.20}$$

$$M_K(u) = \left\{ \int_K u \cdot q dV \text{ for all } q \in R_{k-1} \right\}. \tag{6.21}$$

Here, $R_{k-1} = (P_{k-2})^3 \bigoplus \mathcal{S}_{k-1}$ and $\mathcal{S}_{k-1} = \{v \in \text{homogeneous polynomials of degree } k-1 : \ x \cdot v = 0\}$

Unisolvent

Lemma 41. *The degrees of freedom* (6.20) *and* (6.21) *for u on K vanish if and only if the degrees of freedom for $\hat{u}$ on $\hat{K}$ vanish.*

Lemma 42. *If $u \in (P_k)^3$ and all the degrees of freedom in* (6.20) *for u associated with a given face f vanish, then $u \cdot v = 0$ on f.*

Lemma 43. *If $u \in (P_k)^3$ and all the degrees of freedom in* (6.20) *and* (6.21) *vanish, then $u = 0$.*

Proof. By Lemma 42, $u \cdot v = 0$ on each face f of K. Besides, the normal trace of u on ∂K vanishes, then

$$\int_K \nabla \cdot u q dV = -\int_K u \cdot \nabla q dV = 0 \text{ for all } q \in P_{k-1}, \tag{6.22}$$

so we get $\nabla \cdot u = 0$. We can write $P_{k-1}^3 = R_{k-1} + \nabla \tilde{P}_k$, then

$$\int_K u \cdot q dV = 0 \text{ for all } q \in (P_{k-1})^3. \tag{6.23}$$

Mapping to the reference element, then the fact that $\hat{u} = 0$ on the faces of $\hat{K}$ implies that $u = (\hat{x}_1\phi_1, \hat{x}_2\phi_2, \hat{x}_3\phi_3)$ with $(\phi_1, \phi_2, \phi_3) \in (P_{k-1})^3$. Choosing $q = (\phi_1, \phi_2, \phi_3)^T$ in (6.23), we get $u = 0$. □

The space $H_h^{\text{curl}}(K)$

Definition

For H_h^{curl}, on each element K, the shape function space is $(P_k)^3$, and the degrees of freedoms is composed of three sets. For a function u on K such that $u \in (H^{1/2+s}(K))^3, \delta > 0$ and $\nabla \times u \in (L^q(K))^3, q > 2$, we define

$$M_e(u) = \left\{ \int_f (u \cdot \tau) q ds \text{ for all } q \in P_k(e) \text{ for each edge } e \text{ of } K \right\}, \tag{6.24}$$

$$M_f(u) = \left\{ \int_f u \cdot q dA \text{ for all } q \in D_{k-1}(f) \text{ for each face } f \text{ of } K \right\}, \tag{6.25}$$

$$M_K(u) = \left\{ \int_K u \cdot q dV \text{ for all } q \in D_{k-2}(K) \right\}. \tag{6.26}$$

Here $D_{k-1}(f) = (P_{k-2}(f))^2 \oplus \tilde{P}_{k-2}(f)$.

Unisolvent

Lemma 44. *The degrees of freedom* (6.24)-(6.26) *for a function u on K vanish if and only if they vanish for $\hat{u}$ on $\hat{K}$.*

Lemma 45. *If $u \in (P_k)^3$ and all degrees of freedom associated with a face f and the edges of f vanish. Then $u \times v = 0$ on f.*

Lemma 46. *If $u \in (P_k)^3$ and all degrees of freedom of type* (6.24)-(6.26) *vanish, then $u = 0$.*

Proof. By Lemma 45, $u \times v = 0$ on ∂K. Noticing $\nabla \times q \in D_{k-2}(K)$, and using the three dimensional Stokes formula and the volume degrees of freedom

$$\int_E \nabla \times u \cdot q dA = \int_K u \cdot \nabla \times q dV = 0 \text{ for all } q \in (P_{k-1})^3, \tag{6.27}$$

Selecting $q = \nabla \times u$, we have $\nabla \times u = 0$. Then all degrees of freedom vanish on $\hat{K}$ and hence $\nabla \times \hat{u} = 0$ in $\hat{K}$ and $\hat{u} \times \hat{v} = 0$. Then we can show $\hat{u} = 0$ and hence $u = 0$. □

6.7.3 $L_h^2(K)$

Definition

For $k = 0$ the shape function space for L_h^2 on each element K is 1-dimensional

$$\mathcal{P} = \{\alpha, \alpha \in \mathbb{R}\}$$

and the degrees of freedoms are

$$\mathcal{N} = \left\{p(a_{ijk})\right\}.$$

Unisolvent

When $p(a_{ijk}) = 0$, we get $\alpha = 0$.

Generalization:

L_h^2: arbitrary piecewise polynomials of degree at most k On each tetrahedral element τ, the elements of L_h^2 are arbitrary $\mathcal{P}_k$ and degrees of freedoms are the tetrahedral moments of order at most k.

6.8 Global finite element spaces

6.8.1 Triangulations

The finite element discretization is based on a partition of the domain Ω. For $d = 1$, the partition is a union of intervals. For $d = 2$, the partition is a union of triangles (see the left side of Fig. 6 for a uniform triangulation.

A finite element space is a finite dimensional vector space consisting of piecewise polynomials with respect to a partition of a domain.

Given a bounded tetrahedral domain $\Omega \subset \mathbb{R}^d$. A triangulation of Ω is a special partition of Ω consisting of a set $\mathcal{T}_h$ of $d-$simplices with

$$h_\tau = \text{diam}(\tau), \quad h = \max_{\tau \in \mathcal{T}_h} h_\tau, \quad \underline{h} = \min_{\tau \in \mathcal{T}_h} h_\tau,$$

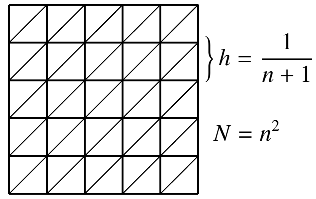

Fig. 6.6. Two-dimensional uniform grid for finite element

such that

1. The intersection of any two simplices in $\mathcal{T}_h$ either consists of a common lower dimensional simplex or is empty;
2.

$$\max_{\tau\in\mathcal{T}_h} \frac{h_\tau}{\rho_\tau} \le \sigma_1 \tag{6.28}$$

where ρ_τ denotes the radius of the ball inscribed in τ.

The triangulation $\{\mathcal{T}_h : h \in \aleph\}$ is said to be quasi-uniform if it satisfies (6.28) and the following

$$h \le \sigma_3 \underline{h}. \tag{6.29}$$

The assumption (6.28) is a local assumption, as is meant by above definition, for $d = 2$ for example, it assures that each triangle will not degenerate into a segment in the limiting case. A triangulation satisfying this assumption is often called to be *shape regular*.

On the other hand, the assumption (6.29) is a global assumption, which says that the smallest mesh size is not too small compared with the largest mesh size of the same triangulation. By the definition, in a quasiuniform triangulation, all the elements are about the same size asymptotically.

In the rest of this chapter, unless otherwise noted, we assume that the triangulation $\{\mathcal{T}_h : h \in \aleph\}$ is quasi-uniform.

6.8.2 Obtaining a global finite element space from local space

Definition 8. *Given a triangulation $\mathcal{T}_h$ of Ω, Given finite element space $(K, \Sigma_K, \mathcal{P}_K)$, a global finite element space V_h consists of functions, v_h, satisfying the following two properties:*

1. *For any $K \in \mathcal{T}_h$, $v_h|_K \in P_K$;*
2. *For any $K, K' \in \mathcal{T}_h$, v_h has the same degrees of freedom on $K \cap K'$.*

6.8.3 Finite element spaces

Corresponding to the triangulations $\mathcal{T}_h$, the finite element spaces $\mathcal{S}^h = \mathcal{S}^h(\Omega)$ is defined by

$$\mathcal{S}^h(\Omega) = \{v \in C(\overline{\Omega}) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}_h\}.$$

It is easy to see that $\mathcal{S}^h(\Omega) \subset H^1(\Omega)$. We further define that

$$\mathcal{S}^h_0(\Omega) = \mathcal{S}^h(\Omega) \cap H^1_0(\Omega).$$

Namely, $\mathcal{S}^h_0$ consists of piecewise linear continuous functions.

Nodal basis function

Definition 9 (Nodal basis function). *Let $(K, \mathcal{P}, \mathcal{N})$ be a finite element triplet. $\{\phi_1, \phi_2, \ldots, \phi_m\}$ is called the nodal basis for $\mathcal{P}$ if it is the basis for $\mathcal{P}$ that is duel to $\mathcal{N}$, namely $N_i(\phi_j) = \delta_{ij}$.*

Let us consider one simple example. Let $\mathcal{N}_h = \{x_i, i = 1, \ldots, N_h\}$ be the set of all interior nodes of $\mathcal{T}_h$ and $\{\phi_1, \ldots, \phi_{n_h}\}$ be the standard piecewise linear nodal basis functions such that ϕ_i is equal to one at precisely one node x_i and vanishs at all other nodal points. Namely

$$\phi_i(x_j) = \delta_{ij}.$$

It is clear that ϕ_i is locally supported. In fact

$$\text{supp } \phi_i = \bigcup_{\tau \in \mathcal{T}_{h, x_i \in \tau}} \tau.$$

Here $\mathcal{T}_{h, x_i \in \tau}$ denotes the triangles contain the edge which belongs to τ.

It is evident that $\mathcal{S}_0^h$ is a linear vector space of dimension N and that each internal nodal point corresponds to a degree of freedom. $\mathcal{S}_0^h$ has a natural basis known as the nodal basis, which consists of the unique set of functions $\{\phi_i\} \subset \mathcal{S}_0^h$ satisfying

$$\phi_i(x_j) = \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & \text{otherwise.} \end{cases} \qquad 1 \le i, j \le N.$$

For $d = 1$, we have for $i = 1, 2, \cdots, N$

$$\phi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i]; \\ \frac{x_{i+1} - x}{h}, & x \in [x_i, x_{i+1}]; \\ 0, & \text{elsewhere.} \end{cases}$$

For $d = 2$, the expression for each ϕ is a little bit involved, but a typical profile is shown in Figure 6.7.

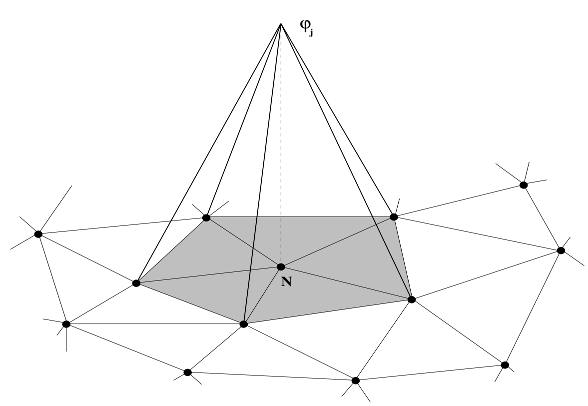

Fig. 6.7. A typical nodal basis function

6.9 Global $H(grad)$, $H(div)$ and $H(curl)$ Finite element spaces

The Sobolev spaces $H(\text{div})$ and $H(\text{curl})$ are two important classes of spaces that are important for mixed formulation of second order elliptic equations and for Maxewell equations in electromagnetic applications. In this section, we give some brief discussions on finite element subspaces for these two spaces and the corresponding multigrid methods.

Given a Lipschitz domain Ω and a linear differential operator B, we define

$$H(\mathrm{B};\Omega) = \{v \in (L^2(\Omega))^n, \mathrm{B}v \in L^2(\Omega)\}.$$

By taking B =div, curl, we obtain the Sobolev spaces $H(\text{div};\Omega)$ and $H(\text{curl};\Omega)$ that we are interested. We also notice that

$$H^1(\Omega) = H(\text{grad};\Omega), L^2(\Omega) = H(0;\Omega).$$

For clarity, details are only given for the descriptions of $H(\text{div};\Omega)$ and $H(\text{curl};\Omega)$ in three dimensions for simplicial elements. Results for two dimensions and other type of elements (such as tensor product elements) will only be listed without proofs.

Let us assume that Ω is a polyhedron in R^3 and it is partitioned by a quasi-uniform grid T_h. Fix an integer k, we introduce below the following finite element spaces for H^1, $H(\text{div})$, $H(\text{curl})$ and L^2 respectively.

6.10 Exact sequences

6.10.1 3D

$$R \longrightarrow H_h^1 \xrightarrow{\text{grad}} H_h^{\text{curl}} \xrightarrow{\text{curl}} H_h^{\text{div}} \xrightarrow{\text{div}} L_h^2 \longrightarrow 0$$

We define $\text{grad}_h : L_h^2 \mapsto H_h^{\text{div}}$ as the adjoint of $-\text{div}$ by

$$(\text{grad}_h u_h, v_h) = -(u_h, \text{div } v_h) \quad u_h \in L_h^2, v_h \in H_h^{\text{div}},$$

and $\text{curl}_h : H_h^{\text{div}} \mapsto H_h^{\text{curl}}$ as the adjoint of $\text{curl} : H_h^{\text{curl}} \mapsto H_h^{\text{div}}$ by

$$(\text{curl}_h v_h, w_h) = (v_h, \text{curl} w_h) \quad v_h \in H_h^{\text{div}}, w_h \in H_h^{\text{curl}}.$$

With these two new operators, we can easily formulate the discrete Helmholtz decomposition. It follows that

$$H_h^{\text{curl}} = \text{kernel(curl)} \oplus \text{range}(\text{curl}^*))$$

But

$$\text{kernel(curl)} = \text{range(grad)} = \text{grad}(H_h^1)$$

and

$$\text{range}(\text{curl}^*) = \text{range}(\text{curl}_h) = \text{curl}_h(H_h^{\text{div}}).$$

Hence we obtain the following decomposition

$$H_h^{\text{curl}} = \text{grad}(H_h^1) \oplus \text{curl}_h(H_h^{\text{div}}) \tag{6.30}$$

which are orthogonal with respect to both L^2 and $H(curl)$ inner products.

Similarly

$$H_h^{\text{div}} = \text{grad}_h(L_h^2) \oplus \text{curl}(H_h^{\text{curl}}) \tag{6.31}$$

which are orthogonal with respect to both L^2 and $H(div)$ inner products.

6.10.2 2D

Similarly, we have the discrete Helmholtz decomposition in 2D. The two exact sequences we have in 2D are: Similar to the 3D case, we define $\mathrm{curl}_h : L_h^2 \mapsto H_h^{\mathrm{rot}}$ as the adjoint operator of rot by

$$\begin{array}{ccccccccc}
R & \longrightarrow & H_h^1 & \xrightarrow{\mathrm{grad}} & H_h^{\mathrm{rot}} & \xrightarrow{\mathrm{rot}} & L_h^2 & \longrightarrow & 0 \\
R & \longrightarrow & H_h^{\mathrm{curl}} & \xrightarrow{\mathrm{curl}} & H_h^{\mathrm{div}} & \xrightarrow{\mathrm{div}} & L_h^2 & \longrightarrow & 0
\end{array}$$

$$(\mathrm{curl}_h u_h, v_h) = (u_h, \mathrm{rot} v_h), \quad u_h \in L_h^2, v_h \in H_h^{\mathrm{rot}},$$

and $\mathrm{grad}_h : L_h^2 \mapsto H_h^{\mathrm{div}}$ by

$$(\mathrm{grad}_h u_h, v_h) = -(u_h, \mathrm{div} v_h), \quad u_h \in L_h^2, v_h \in H_h^{\mathrm{div}}.$$

Therefore,

$$H_h^{\mathrm{rot}} = \mathrm{kernel(rot)} \oplus \mathrm{range(rot^*)}$$

Since

$$\mathrm{kernel(rot)} = \mathrm{range(grad)} = \mathrm{grad}(H_h^1),$$

and

$$\mathrm{range(rot^*)} = \mathrm{range}(\mathrm{curl}_h) = \mathrm{curl}_h(L_h^2),$$

we get

$$H_h^{\mathrm{rot}} = \mathrm{grad}(H_h^1) \oplus \mathrm{curl}_h(L_h^2).$$

Similarly, we get

$$H_h^{\mathrm{div}} = \mathrm{curl}(H_h^{\mathrm{curl}}) \oplus \mathrm{grad}_h(L_h^2).$$

6.11 Interpolants, exact sequencies and commutative diagrams

Let $\Pi_h, \Pi_h^{\mathrm{curl}}, \Pi_h^{\mathrm{div}}$ and Π_h^0 be the interpolants associated with the degrees of freedom as described above for H_h^1, H_h^{curl}, H_h^{div} and L_h^2 respectively. We note that, by definition, Π_h^0 is just the L^2 projection onto L_h^2. The intimate relationships between the spaces $H_h^1, H_h^{\mathrm{curl}}, H_h^{\mathrm{div}}$ and L_h^2 are illustrated by the fact that the following two sequences are exact and the underlying three diagrams are all commutative:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & C^\infty & \xrightarrow{\mathrm{grad}} & C^\infty & \xrightarrow{\mathrm{curl}} & C^\infty & \xrightarrow{\mathrm{div}} & C^\infty & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \Pi_h} & & \downarrow{\scriptstyle \Pi_h^{\mathrm{curl}}} & & \downarrow{\scriptstyle \Pi_h^{\mathrm{div}}} & & \downarrow{\scriptstyle \Pi_h^0} & & \\
R & \longrightarrow & H_h^1 & \xrightarrow{\mathrm{grad}} & H_h^{\mathrm{curl}} & \xrightarrow{\mathrm{curl}} & H_h^{\mathrm{div}} & \xrightarrow{\mathrm{div}} & L_h^2 & \longrightarrow & 0
\end{array}$$

Fig. 6.8. Exact sequences and commutative diagrams

There are a lot that can be said about the above two exact sequencies and three commutative diagrams. For example, the second (discrete) exact sequence implies that

(6.32) $$\ker(\mathrm{curl}) = \mathrm{grad} H_h^1, \quad \ker(\mathrm{div}\ ) = \mathrm{curl} H_h^{\mathrm{curl}}.$$

6.12 Domains for the cananical interpolants and error estimates

We shall give a brief discussions on the interpolants $\Pi_h, \Pi_h^{\text{curl}}, \Pi_h^{\text{div}}$ and Π_h^0. It is well-known that Π_h is not well defined on H^1 (except in one dimension), but it is well-defined on continuous functions. Π_h^0 is the standard L^2 projection and it is defined on all L^2 function.

For Π_h^{div}, we have

Lemma 47. *Π_h^{div} is well-defined on H^1 and we have*

$$\|v - \Pi_h^{\text{div}} v\| \lesssim h|v|_1, \quad v \in H^1. \tag{6.33}$$

Because of the dependence on edge moments, the situation is more complicated for the operator Π_h^{curl}. It is easy to see that it is not well-defined on H^1, but it can be proved that it is well-defined and bounded on a subspace of H^1 vector fields whose curl belongs to L^p for any fixed $p > 2$ (see Amrouche, Bernardi, Dauge and Girault (1998)). In particular, it is a bounded operator on H^1 vector fields whose curl is piecewise polynomial (say, in H_h^{div}).

Lemma 48.

$$\|v - \Pi_h^{\text{curl}} v\| \lesssim h|v|_1, \quad v \in H^1 \textit{ such that } \nabla \times v \in H_h^{\text{div}}. \tag{6.34}$$

Proof. To show this, let us first consider the case where the mesh consists of only the unit reference simplex $\hat{K}$. In this case

$$\|\Pi_h^{\text{curl}} v\| \lesssim (\|v\|_1 + \|\nabla v\|_{L^\infty}) \lesssim (\|v\|_1 + \|\nabla v\|_{L^2}) \lesssim \|v\|_1.$$

The desired error estimate then follows by a standard Bramble-Hilbert scaling argument. □

Since $H^1(\Omega) \hookrightarrow L^6(\Omega)$ in three dimensions, we have

Lemma 49. *Π_h^{curl} is well defined on function $v \in [H^1(\Omega)]^3$ such that* curl$v \in [H^1(\Omega)]^3$ *and*

$$\|v - \Pi_h^{\text{curl}} v\| \lesssim h(|v|_1 + |\text{curl} v|_1)$$

6.13 Exercise

Exercise 1. Show that $\mathcal{S}^h$ is indeed a subspace of $H^1(\Omega)$.

Exercise 2. Assume that $\partial\Omega$ is curved but piecewise convex. Define

$$\mathcal{S}^h = \{v : v \in S^h(\bar{\Omega}_h), v|_{\Omega\backslash\Omega_h} \text{ is the natural linear extention}\} \tag{6.35}$$

and

$$\mathcal{S}_0^h = \{v : v \in S_0^h(\bar{\Omega}_h), v|_{\Omega\backslash\Omega_h} = 0\}. \tag{6.36}$$

Prove that $\mathcal{S}^h \subset H^1(\Omega)$ and $\mathcal{S}_0^h \subset H_0^1(\Omega)$.

Exercise 3. Prove, for the space $\mathcal{S}^h$ defined in the above Exercise,

$$\|v\|_{0,\Omega\backslash\Omega_h} \lesssim h\|v\|_1 \quad \forall\, v \in \mathcal{S}^h.$$