

A Posteriori Error Estimates and Adaptive Finite Element Methods

7.1 Convergence of residual type a posteriori error estimates for the Poisson equation

Let Ω be a polygonal bounded domain in \mathbb{R}^d for $d = 2, 3$, and let u be the solution to the following problem:

$$(7.1) \quad -\Delta u = f \quad \text{in } \Omega,$$

$$(7.2) \quad u = 0 \quad \text{on } \partial\Omega,$$

where $f \in L^2(\Omega)$.

Define $H_0^1(\Omega)$ to be the space of functions in $H^1(\Omega)$ with zero boundary values, then the weak formulation of (7.1)-(7.2) is to find $u \in H_0^1(\Omega)$ such that

$$(7.3) \quad (\nabla u, \nabla v) = (f, v) \quad \forall v \in H_0^1(\Omega).$$

Denote V_h^r as the finite element space of all continuous functions which are piecewise polynomial of degree r on each element, and V_{0h}^r as the space of functions in V_h^r with zero boundary value, then the finite element approximation of (7.3) is to find $u_k \in V_{0h}^r \subset H_0^1(\Omega)$ such that

$$(7.4) \quad (\nabla u_H, \nabla v) = (f, v) \quad \forall v \in V_{0h}^r.$$

Denote \mathcal{T}_H and \mathcal{E}_H as the set of elements and edges respectively. Let u_H be the solution of (7.4), subtracting (7.4) from (7.3), then for any $v \in H_0^1(\Omega)$, we have

$$(7.5) \quad \begin{aligned} (\nabla(u - u_H), \nabla v) &= \sum_{T \in \mathcal{T}_H} (f + \Delta u_H, v - \mathcal{I}_H v)_T - \sum_{E \in \mathcal{E}_H} \int_E [\nabla u_H \cdot \mathbf{n} (v - \mathcal{I}_H v)] dS \\ &= \sum_{T \in \mathcal{T}_H} (f + \Delta u_H, v - \mathcal{I}_H v)_T - \sum_{E \in \mathcal{E}_H} \langle [\nabla u_H] \cdot \mathbf{n}, \{v - \mathcal{I}_H v\} \rangle_E \end{aligned}$$

where \mathcal{I}_H is the Scott-Zhang quasi interpolation operator.

In this chapter, we use C to denote a generic constant. Besides, we list all the notations which will be used in this chapter below:

ω_T : the set of mesh elements which share at least one edge with T ;

$\tilde{\omega}_T$: the set of mesh elements which share at least one point with T ;

ω_E : the set of mesh elements which share at least one edge with E ;

$\tilde{\omega}_E$: the set of mesh elements which share at least one point with E ;
 \mathcal{E}_T : the set of edges of T ;
 H_T : the size of the mesh element T ;
 \mathcal{N}_T : the vertices of all elements in \mathcal{T} .

Then define the local error indicator associated with mesh element T :

$$(7.6) \quad \eta_T^2 := \sum_{E \in \mathcal{E}_T} \|H_T^{\frac{1}{2}} [\nabla u_H \cdot \mathbf{n}]\|_E^2 + \|H_T (f + \Delta u_H)\|_T^2,$$

and for subset $\tilde{\mathcal{T}}_H \subset \mathcal{T}_H$, define

$$(7.7) \quad \eta^2(u_H, \tilde{\mathcal{T}}_H) := \sum_{T \in \tilde{\mathcal{T}}_H} \eta_T^2.$$

Taking $v = u - u_H$ in (7.5), we can prove the upper bound (reliability) and the lower bound (efficiency) of the error.

Theorem 53. (Reliability) Define f_T as the mean of f over triangle T , then

$$(7.8) \quad \|u - u_H\|_{H^1(\Omega)}^2 \leq C \eta^2(u_H, \mathcal{T}_H).$$

Proof. By the Cauchy-Schwarz inequality, we have $\forall v \in H_0^1(\Omega)$,

$$(7.9) \quad |v|_{H^1(\Omega)} = \sup_{\substack{w \in H_0^1(\Omega), \\ |w|_{H^1(\Omega)}=1}} \int_{\Omega} \nabla v \nabla w dx.$$

Combine (7.5) and (7.9), we only need to estimate the supremum of the right-hand side of (7.5). Using Cauchy-Schwarz inequality and properties of Scott-Zhang quasi interpolation, we have

$$(7.10) \quad \begin{aligned} & \sum_{T \in \mathcal{T}_H} (f + \Delta u_H, w - \mathcal{I}_H w)_T - \sum_{E \in \mathcal{E}_H} \langle [\nabla u_H] \cdot \mathbf{n}, \{w - \mathcal{I}_H w\} \rangle_E \\ & \leq C \left\{ \sum_{T \in \mathcal{T}_H} H_T^2 \|f + \Delta u_H\|_T^2 + \sum_{E \in \mathcal{E}_H} H_T \|[\nabla u_H]\|_E^2 \right\}^{\frac{1}{2}} \\ & \quad \cdot \left\{ \sum_{T \in \mathcal{T}_H} |w|_{1, \tilde{w}_T}^2 + \sum_{E \in \mathcal{E}_H} |w|_{1, \tilde{w}_E}^2 \right\}^{\frac{1}{2}} \\ & \leq C \left\{ \sum_{T \in \mathcal{T}_H} H_T^2 \|f + \Delta u_H\|_T^2 + \sum_{E \in \mathcal{E}_H} H_T \|[\nabla u_H]\|_E^2 \right\}^{\frac{1}{2}} |w|_{H^1(\Omega)}. \end{aligned}$$

Then the upper bound of the error (7.8) is obtained. \square

Before the convergence is given, two lemmas need to be proved in the following. Denote u_k and u_{k+1} as the solutions on k and $k + 1$ bisection levels, \mathcal{T}_k and \mathcal{T}_{k+1} as the meshes on k and $k + 1$ bisection levels respectively, then we have:

Lemma 50. Define $\rho = 1 - \frac{1}{\sqrt{2}}$, then

$$\eta^2(u_k, \mathcal{T}_{k+1}) \leq \eta^2(u_k, \mathcal{T}_k) - \rho \eta^2(u_k, \mathcal{T}_k \setminus \mathcal{T}_{k+1}).$$

Proof.

$$(7.11) \quad \eta^2(u_k, \mathcal{T}_{k+1}) = \eta^2(u_k, \mathcal{T}_{k+1} \cap \mathcal{T}_k) + \eta^2(u_k, \mathcal{T}_{k+1} \setminus \mathcal{T}_k).$$

Assume T is subdivided into $T = T_1 \cup T_2$ with $T_1, T_2 \in \mathcal{T}_{k+1}$ and $|T_1| = |T_2| = \frac{1}{2}|T|$. It is easy to show that

$$(7.12) \quad \sum_{i=1}^2 \eta_{T_i}^2(u_k) \leq \frac{1}{\sqrt{2}} \eta_T^2(u_k),$$

then we have

$$(7.13) \quad \sum_{T \in \mathcal{T}_{k+1} \setminus \mathcal{T}_k} \sum_{i=1}^2 \eta_{T_i}^2(u_k) \leq \frac{1}{\sqrt{2}} \eta^2(u_k, \mathcal{T}_k \setminus \mathcal{T}_{k+1}).$$

Plug (7.13) into (7.11), then the lemma is proved. \square

Lemma 51. *Suppose θ to be the constant in the bulk criterion (7.18), for any $\epsilon > 0$, we have*

$$(7.14) \quad |\eta_{\mathcal{T}}(u_{k+1}, \mathcal{T}_{k+1}) - \eta_{\mathcal{T}}(u_k, \mathcal{T}_{k+1})| \leq C \|\nabla(u_{k+1} - u_k)\|.$$

Proof. Using the definition of the estimator, we have

$$(7.15) \quad \begin{aligned} & |\eta_{\mathcal{T}}(u_{k+1}, \mathcal{T}_{k+1}) - \eta_{\mathcal{T}}(u_k, \mathcal{T}_{k+1})| \\ & \leq C \left| \left(\sum_{T \in \mathcal{T}_{k+1}} \|H_T(f + \Delta u_{k+1})\|_T^2 \right)^{\frac{1}{2}} - \left(\sum_{T \in \mathcal{T}_{k+1}} \|H_T(f + \Delta u_k)\|_T^2 \right)^{\frac{1}{2}} \right| \\ & \quad + C \left| \left(\sum_{T \in \mathcal{T}_{k+1}} \sum_{E \in \partial T} H_T \|[\nabla u_{k+1} \cdot \mathbf{n}]\|_E^2 \right)^{\frac{1}{2}} - \left(\sum_{T \in \mathcal{T}_{k+1}} \sum_{E \in \partial T} H_T \|[\nabla u_k \cdot \mathbf{n}]\|_E^2 \right)^{\frac{1}{2}} \right| \\ & := S_1 + S_2. \end{aligned}$$

Using the trace inequality, the first term can be bounded by

$$(7.16) \quad \begin{aligned} S_1 & \leq C \left(\sum_{T \in \mathcal{T}_{k+1}} H_T^2 \|\Delta(u_{k+1} - u_k)\|_T^2 \right)^{\frac{1}{2}} \\ & \leq C \|\nabla(u_{k+1} - u_k)\|. \end{aligned}$$

Using the trace inequality, the second term can be bounded by

$$(7.17) \quad \begin{aligned} S_2 & \leq C \left(\sum_{T \in \mathcal{T}_{k+1}} \sum_{E \in \partial T} H_T \|[\nabla(u_{k+1} - u_k) \cdot \mathbf{n}]\|_E^2 \right)^{\frac{1}{2}} \\ & \leq C \|\nabla(u_{k+1} - u_k)\|. \end{aligned}$$

The theorem is proved by combining (7.16) and (7.17). \square

Lemma 52. *Assume there exists $\theta < 1$ such that the following bulk criterion holds*

$$(7.18) \quad \eta^2(u_k, \mathcal{T}_k \setminus \mathcal{T}_{k+1}) \geq \theta \eta^2(u_k, \mathcal{T}_k),$$

then for any $\epsilon > 0$, we have

$$(7.19) \quad \eta^2(u_{k+1}, \mathcal{T}_{k+1}) \leq (1 + \epsilon)(1 - \rho\theta)\eta^2(u_k, \mathcal{T}_k) + \frac{1}{\beta_2(\epsilon)} \|\nabla(u_{k+1} - u_k)\|^2.$$

Proof. By Lemma 51, we have

$$(7.20) \quad \eta_{\mathcal{T}}(u_{k+1}, \mathcal{T}_{k+1}) \leq \eta_{\mathcal{T}}(u_k, \mathcal{T}_{k+1}) + C\|\nabla(u_{k+1} - u_k)\|.$$

Then using (7.18), Lemma 50, and the Young's inequality, we get

$$(7.21) \quad \begin{aligned} \eta^2(u_{k+1}, \mathcal{T}_{k+1}) &\leq (1 + \epsilon)\eta^2(u_k, \mathcal{T}_k) + \frac{1}{\beta_2(\epsilon)} \|\nabla(u_{k+1} - u_k)\|^2 \\ &\quad - (1 + \epsilon)\rho\eta^2(u_k, \mathcal{T}_k \setminus \mathcal{T}_{k+1}), \\ &\leq (1 + \epsilon)\eta^2(u_k, \mathcal{T}_k) + \frac{1}{\beta_2(\epsilon)} \|\nabla(u_{k+1} - u_k)\|^2 \\ &\quad - (1 + \epsilon)\rho\theta\eta^2(u_k, \mathcal{T}_k) \\ &\leq (1 + \epsilon)(1 - \rho\theta)\eta^2(u_k, \mathcal{T}_k) + \frac{1}{\beta_2(\epsilon)} \|\nabla(u_{k+1} - u_k)\|^2, \end{aligned}$$

where $\beta_2(\epsilon) = \frac{C\epsilon}{\epsilon+1}$. \square

Next we state the convergence theorem,

Theorem 54. *The following error contraction property holds*

$$\|\nabla(u - u_{k+1})\|^2 + \beta_2(\epsilon)\eta^2(u_{k+1}, \mathcal{T}_{k+1}) \leq \alpha(\|\nabla(u - u_k)\|^2 + \beta_2(\epsilon)\eta^2(u_k, \mathcal{T}_k)),$$

where $\alpha = \frac{C+(1+\epsilon)(1-\rho\theta)\beta_2(\epsilon)}{C+\beta_2(\epsilon)}$.

Proof. By orthogonality, we have

$$(7.22) \quad \|\nabla(u - u_{k+1})\|^2 = \|\nabla(u - u_k)\|^2 - \|\nabla(u_k - u_{k+1})\|^2,$$

Using Lemma 52, we have

$$(7.23) \quad \begin{aligned} \|\nabla(u - u_{k+1})\|^2 + \beta_2(\epsilon)\eta^2(u_{k+1}, \mathcal{T}_{k+1}) \\ \leq \|\nabla(u - u_k)\|^2 + (1 + \epsilon)(1 - \rho\theta)\beta_2(\epsilon)\eta^2(u_k, \mathcal{T}_k). \end{aligned}$$

Using the reliability of the estimator, we have

$$(7.24) \quad \|\nabla(u - u_{k+1})\|^2 + \beta_2(\epsilon)\eta^2(u_{k+1}, \mathcal{T}_{k+1}) \leq \alpha(\epsilon)(\|\nabla(u - u_k)\|^2 + \beta_2(\epsilon)\eta^2(u_k, \mathcal{T}_k)),$$

where $\alpha(\epsilon) = \frac{C+(1+\epsilon)(1-\rho\theta)\beta_2(\epsilon)}{C+\beta_2(\epsilon)}$, which is less than 1 when ϵ becomes small. Then the theorem is proved by iteration. \square

The following result is a direct corollary of Theorem 54.

Corollary 6. *The following error reduction property holds*

$$\|\nabla(u - u_{k+1})\|^2 + \beta_2(\epsilon)\eta^2(u_{k+1}, \mathcal{T}_{k+1}) \leq \alpha^{k+1}(\|\nabla(u - u_0)\|^2 + \beta_2(\epsilon)\eta^2(u_0, \mathcal{T}_0)).$$

7.2 Convergence rate of residual type a posteriori error estimates for the Poisson equation

Next we prove the lower bound of the error.

Theorem 55. (Efficiency) Define f_T as the mean of f over triangle T , then

$$(7.25) \quad C\eta_T^2 - C \sum_{T' \in \mathcal{N}_T} H_{T'}^2 \|f - f_{T'}\|_{T'}^2 \leq C \|\nabla(u - u_H)\|_{w_E}^2.$$

Proof. Fix an arbitrary element T and choose the following function as a test function

$$(7.26) \quad w_T = (f_T + \Delta u_H) \psi_T,$$

Here ψ_T is a element bubble function

$$(7.27) \quad \psi_T = \alpha_T \prod_{x \in T \cap \mathcal{N}_T} \lambda_x,$$

where α_T is a constant depending on the meshes.

Then we get

$$(7.28) \quad \int_T (f + \Delta u_H) w_T dx = \int_T \nabla(u - u_H) \cdot \nabla w_T dx.$$

Adding $\int_T (f_T - f) w_T dx$ to both sides of (7.28), we get

$$(7.29) \quad \begin{aligned} \int_T (f_T + \Delta u_H)^2 \psi_T dx &= \int_T (f_T + \Delta u_H) w_T dx \\ &= \int_T \nabla(u - u_H) \cdot \nabla w_T dx - \int_T (f - f_T) w_T dx. \end{aligned}$$

By using the following properties of bubble functions,

$$(7.30) \quad \int_K (f_T + \Delta u_H)^2 \psi_T dx \geq C \|f_T + \Delta u_H\|_T^2,$$

$$(7.31) \quad \int_T \nabla(u - u_H) \cdot \nabla w_T dx \leq C \|\nabla(u - u_H)\|_T H_T^{-1} \|f_T + \Delta u_H\|_T,$$

$$(7.32) \quad \int_T (f - f_T) w_T dx \leq C \|f - f_T\|_T \|f_T + \Delta u_H\|_T,$$

we obtain

$$(7.33) \quad H_T \|f_T + \Delta u_H\|_T \leq C \|\nabla(u - u_H)\|_T + C H_T \|f - f_T\|_T.$$

Next consider the interior edge E and choose the following function as a test function

$$(7.34) \quad w_E = [\nabla u_H \cdot \mathbf{n}] \psi_E,$$

and here ψ_E is a bubble function

$$(7.35) \quad \psi_E = \beta_E \prod_{x \in E \cap \mathcal{N}_T} \lambda_x,$$

here β_E is a constant depends on the meshes.

Then we have

$$(7.36) \quad \begin{aligned} \int_E [\nabla u_H \cdot \mathbf{n}]^2 \psi_E dx &= \int_{w_E} \nabla(u - u_H) \cdot \nabla w_E dx + \int_{w_E} (f + \Delta u_H) w_E dx \\ &= \int_{w_E} \nabla(u - u_H) \cdot \nabla w_E dx + \int_{w_E} (f_T + \Delta u_H) w_E dx + \int_{w_E} (f - f_T) w_E dx. \end{aligned}$$

Using the following properties of bubble functions,

$$(7.37) \quad \int_E [\nabla u_H \cdot \mathbf{n}]^2 \psi_E ds \geq C \|\nabla u_H \cdot \mathbf{n}\|_E^2,$$

$$(7.38) \quad \int_{w_E} \nabla(u - u_H) \cdot \nabla w_E dx \leq C \|\nabla(u - u_H)\|_{w_E} H_T^{-\frac{1}{2}} \|\nabla u_H \cdot \mathbf{n}\|_E,$$

$$(7.39) \quad \sum_{T \in w_E} \int_T (f_T + \Delta u_H) w_E dx \leq C \sum_{T \in \mathcal{T}_H, E \in \mathcal{E}_T} \|f_T + \Delta u_H\|_T H_T^{\frac{1}{2}} \|\nabla u_H \cdot \mathbf{n}\|,$$

$$(7.40) \quad \sum_{T \in w_E} \int_T (f - f_T) w_E dx \leq \sum_{T \in \mathcal{T}_H, E \in \mathcal{E}_T} \|f - f_T\|_T \|w_E\|_T,$$

and combining (7.36), we get

$$(7.41) \quad \begin{aligned} H_T^{\frac{1}{2}} \|\nabla u_H \cdot \mathbf{n}\|_E &\leq C \|\nabla(u - u_H)\|_{w_E} + C H_T \sum_{T \in \mathcal{T}_H, E \in \mathcal{E}_T} \|f - f_T\|_T \\ &\quad + C H_T \sum_{T \in \mathcal{T}_H, E \in \mathcal{E}_T} \|f_T + \Delta u_H\|_T \\ &\leq C \|\nabla(u - u_H)\|_{w_E} + C H_T \sum_{T \in \mathcal{T}_H, E \in \mathcal{E}_T} \|f - f_T\|_T, \end{aligned}$$

where (7.33) is used in the second inequality.

Taking the summation of (7.41) over all boundaries of T yields the lower bound of the error in (7.25). \square

We can see the bulk condition is very important, so we impose a reasonable condition to make sure the bulk condition is satisfied. For simplicity, we assume the linear element is used. For the higher order element, we just need to define a new oscillation term which contains the jump term and the residue term, and incorporates these terms into the lower bound estimate of the error.

Define the following oscillations by

$$(7.42) \quad \text{osc}_{\mathcal{T}}^2(u_H, T) := H_T^2 \|f - \Delta u_H - (f - \Delta u_H)_T\|_T^2 = H_T^2 \|(f - f_T)\|_T^2,$$

$$(7.43) \quad \text{osc}_{\mathcal{T}_H}^2(u_H, \mathcal{T}_H) := \sum_{T \in \mathcal{T}_H} \text{osc}_{\mathcal{T}}^2(u_H, T).$$

Here f_T and $(f - \Delta u_H)_T$ denote the mean values of f and $f - \Delta u_H$ over T respectively. The subscript \mathcal{T} of osc means $u_H \in \mathcal{T}$.

Based on the upper bound, the lower bound of the error, and the fact that

$$(7.44) \quad \text{osc}_{\mathcal{T}_H}^2(u_H, \mathcal{T}_H) \leq \eta^2(u_H, \mathcal{T}_H),$$

we have

$$(7.45) \quad \begin{aligned} \|u - u_H\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_H}^2(u_H, \mathcal{T}_H) &\leq C\eta^2(u_H, \mathcal{T}_H), \\ \eta^2(u_H, \mathcal{T}_H) &\leq C\|u - u_H\|_{H^1(\Omega)}^2 + C \text{osc}_{\mathcal{T}_H}^2(u_H, \mathcal{T}_H). \end{aligned}$$

Then we approximately have

$$(7.46) \quad \eta^2(u_H, \mathcal{T}_H) \approx \|u - u_H\|_{H^1(\Omega)}^2 + \text{osc}^2(u_H, \mathcal{T}_H).$$

The right-hand side of (7.46) is called total error. The proof of the following lemma can be found in [11]. Note that we can prove it similarly, even the definition of the oscillation is not exactly the same. Denote $V_{\mathcal{T}}$ as the space of functions defined on mesh \mathcal{T} .

Lemma 53. *Assume \mathcal{T}_* is the refinement of the \mathcal{T} , $v \in V_{\mathcal{T}}$ and $v_* \in V_{\mathcal{T}_*}$, then we have*

$$\text{osc}_{\mathcal{T}}^2(v, \mathcal{T} \cap \mathcal{T}_*) \leq 2\text{osc}_{\mathcal{T}_*}^2(v_*, \mathcal{T} \cap \mathcal{T}_*) + C\|v - v_*\|_{H^1(\Omega)}^2.$$

Proof. Using (3.2) in [11] and the fact that $\text{osc}_{\mathcal{T}}^2(v, \mathcal{T}) = \text{osc}_{\mathcal{T}_*}^2(v, \mathcal{T})$, we have

$$(7.47) \quad \text{osc}_{\mathcal{T}}^2(v, T) \leq 2\text{osc}_{\mathcal{T}_*}^2(v_*, T) + C\|v - v_*\|_{H^1(\omega_T)}^2$$

Taking the summation of (7.47) with respect to T over $\mathcal{T} \cap \mathcal{T}_*$ yields the conclusion. \square
The lemma below gives the quasi-optimality of the total error.

Lemma 54. *There exists a constant C which depends only on shape regularity of \mathcal{T}_0 such that*

$$\|u - u_H\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(u_H, \mathcal{T}) \leq C \inf_{v \in V_{\mathcal{T}}} \left(\|u - v\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(v, \mathcal{T}) \right).$$

Proof. $\forall \epsilon > 0$, choosing $v_{\epsilon} \in V_{\mathcal{T}}$ satisfying

$$(7.48) \quad \|u - v_{\epsilon}\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(v_{\epsilon}, \mathcal{T}) \leq (1 + \epsilon) \inf_{v \in V_{\mathcal{T}}} \left(\|u - v\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(v, \mathcal{T}) \right).$$

Setting $\mathcal{T}_* = \mathcal{T}$, $v = u_H$ and $v_* = v_{\epsilon}$ in Lemma 53, we have

$$(7.49) \quad \text{osc}_{\mathcal{T}}^2(u_H, \mathcal{T}) \leq 2\text{osc}_{\mathcal{T}}^2(v_{\epsilon}, \mathcal{T}) + C\|u_H - v_{\epsilon}\|_{H^1(\Omega)}^2.$$

Since $v_{\epsilon}, v \in V_{\mathcal{T}}$, then by Galerkin orthogonality:

$$(7.50) \quad \|u - u_H\|_{H^1(\Omega)}^2 + \|u_H - v_{\epsilon}\|_{H^1(\Omega)}^2 = \|u - v_{\epsilon}\|_{H^1(\Omega)}^2,$$

then together with the Poincaré's inequality, we have

$$(7.51) \quad \begin{aligned} \|u - u_H\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(u_H, \mathcal{T}) &\leq C\|u - v_{\epsilon}\|_{H^1(\Omega)}^2 + 2\text{osc}_{\mathcal{T}}^2(v, \mathcal{T}) \\ &\leq C \inf_{v \in V_{\mathcal{T}}} \left(\|u - v\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(v, \mathcal{T}) \right). \end{aligned}$$

\square

Then it's ready to define an approximation class A_S based on the total error. Let $\mathbb{T}_N \subset \mathbb{T}$ be the set of all possible conforming triangulations generated from \mathcal{T}_0 with at most N more elements comparing with \mathcal{T}_0 , then the quality of the best approximation to the total error is defined by

$$(7.52) \quad \mathbb{E}(N, u, f) := \inf_{\mathcal{T} \in \mathbb{T}_N} \inf_{v \in V_{\mathcal{T}}} (\|u - v\|_{H^1(\Omega)}^2 + \text{osc}^2(v, \mathcal{T}))^{1/2},$$

and for $s > 0$, we define the nonlinear approximation class \mathbb{A}_s by

$$(7.53) \quad \mathbb{A}_s = \left\{ (u, f) \mid |u, f|_s := \sup_{N>0} (N^s \mathbb{E}(N, u, f)) < \infty \right\}.$$

Then it is easy to see that by Lemma 54 the discrete solution satisfies

$$(7.54) \quad \mathbb{E}(N, u, f) \approx \inf_{\mathcal{T} \in \mathbb{T}_N} \{ \eta(u_H, \mathcal{T}) : u_H \text{ is the discrete solution of (7.1)-(7.2)} \}.$$

The following lemma states the fact between the H^1 norm of different discrete solutions and the estimator. Using Lemma 3.6 in [11], we can prove the following lemma easily.

Lemma 55. *Let $U_* \in V_{\mathcal{T}_*}$ and $U \in V_{\mathcal{T}}$ be the discrete solutions, and let \mathcal{R} be the set of refined meshes, then we have*

$$\|U_* - U\|_{H^1(\Omega)}^2 \leq C\eta^2(U, \mathcal{R}).$$

Then we prove the bulk condition holds under some reasonable conditions.

Lemma 56. *(optimal marking) Assume \mathcal{T}_* is the refinement of \mathcal{T} , and $U \in V(\mathcal{T})$ and $U_* \in V(\mathcal{T}_*)$ are two corresponding discrete solutions satisfying*

$$(7.55) \quad \|u - U_*\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T}_*) \leq \mu \left(\|u - U\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(U, \mathcal{T}) \right),$$

then there holds the bulk condition

$$(7.56) \quad \eta(U, \mathcal{R}) \geq \theta \eta(U, \mathcal{T}).$$

Proof. By (7.55) and the efficiency of the error, we have

$$(7.57) \quad \begin{aligned} & (1 - 2\mu)C\eta^2(U, \mathcal{T}) \\ & \leq (1 - 2\mu)(\|u - U\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(U, \mathcal{T})) \\ & \leq \|u - U\|_{H^1(\Omega)}^2 - \|u - U_*\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}}^2(U, \mathcal{T}) - 2\text{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T}_*). \end{aligned}$$

Setting $v = U$ and $v_* = U_*$ in Lemma 53, we have

$$(7.58) \quad \text{osc}_{\mathcal{T}}^2(U, \mathcal{T} \cap \mathcal{T}_*) \leq 2\text{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T} \cap \mathcal{T}_*) + C\|U - U_*\|_{H^1(\Omega)}^2.$$

Noticing the following facts by orthogonality and Lemma 55:

$$(7.59) \quad |u - U|_{H^1(\Omega)}^2 - |u - U_*|_{H^1(\Omega)}^2 = |U - U_*|_{H^1(\Omega)}^2 \leq C\eta^2(U, \mathcal{R}),$$

$$(7.60) \quad \text{osc}_{\mathcal{T}}^2(U, \mathcal{T}) \leq \eta^2(U, \mathcal{T}).$$

Combining (7.58)-(7.60), we have

$$(7.61) \quad \begin{aligned} \text{osc}_{\mathcal{T}}^2(U, \mathcal{T}) - 2\text{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T}_*) & \leq \eta^2(U, \mathcal{R}) + C\|U - U_*\|_{H^1(\Omega)}^2 \\ & \leq C\eta^2(U, \mathcal{R}). \end{aligned}$$

Then we have

$$(7.62) \quad \eta^2(U, \mathcal{R}) \geq (1 - 2\mu)C\eta^2(U, \mathcal{T}) := \theta^2\eta^2(U, \mathcal{T}).$$

□

The upper bound of the cardinality of the marked set \mathcal{M}_k is discussed in the following lemma.

Lemma 57. *Let $\{\mathcal{T}_k, V_k, U_k\}_{k \geq 0}$ be the sequence of meshes, finite element spaces, the discrete solutions. If $(u, f) \in \mathbb{A}_s$, then*

$$(7.63) \quad \#\mathcal{M}_k \leq \left(\frac{C}{\mu}\right)^{\frac{1}{2s}} |u, f|_s^{\frac{1}{s}} \left\{ \|u - U_k\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_k}^2(U_k, \mathcal{T}_k) \right\}^{-\frac{1}{2s}}.$$

Proof. Setting $\epsilon^2 := C^{-1} \mu \left(\|u - U_k\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_k}^2(U_k, \mathcal{T}_k) \right)$ with C defined in Lemma 54. Since $(u, f) \in \mathbb{A}_s$, then there exist a $\mathcal{T}_\epsilon \in \mathbb{T}$ and a $v_\epsilon \in V_{\mathcal{T}_\epsilon}$ such that

$$(7.64) \quad \#\mathcal{T}_\epsilon - \#\mathcal{T}_0 \leq C |u, f|_s^{1/s} \epsilon^{-1/s},$$

$$(7.65) \quad \|u - v_\epsilon\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_\epsilon}^2(v_\epsilon, \mathcal{T}_\epsilon) \leq \epsilon^2.$$

Let \mathcal{T}_* be the overlay of \mathcal{T}_ϵ and \mathcal{T}_k , then

$$(7.66) \quad \begin{aligned} \|u - U_*\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_*}^2(U_*, \mathcal{T}_*) &\leq C \left(\|u - v_\epsilon\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_\epsilon}^2(v_\epsilon, \mathcal{T}_\epsilon) \right) \\ &\leq \mu \left(\|u - U_k\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_k}^2(U_k, \mathcal{T}_k) \right). \end{aligned}$$

Then it satisfies the condition in lemma 56, we can select a subset $\mathcal{M}_k \subset \mathcal{T}_k$ such that

$$(7.67) \quad \#\mathcal{M}_k \leq \#\mathcal{R} \leq \#\mathcal{T}_* - \#\mathcal{T}_k \leq \#\mathcal{T}_\epsilon - \#\mathcal{T}_0,$$

where the property of overlay meshes in Lemma 3.7 in [11] is used in the last inequality.

Combining (7.64) and (7.67), the lemma is derived. \square

To the end, we prove the quasi optimality of the total error.

Theorem 56. *Let $\{\mathcal{T}_k, V_k, U_k\}_{k \geq 0}$ be the sequence of meshes, finite element spaces, the discrete solutions. If $(u, f) \in \mathbb{A}_s$, then*

$$(7.68) \quad \left\{ \|u - U_k\|_{\Omega}^2 + \text{osc}_{\mathcal{T}_k}^2(U_k, \mathcal{T}_k) \right\}^{1/2} \leq C |u, f|_s (\#\mathcal{T}_k - \#\mathcal{T}_0)^{-s}.$$

Proof. By (7.67), we get

$$(7.69) \quad \#\mathcal{T}_k - \#\mathcal{T}_0 \leq C \sum_{j=0}^{k-1} \#\mathcal{M}_j \leq C |u, f|_s^{\frac{1}{s}} \sum_{j=0}^{k-1} \left\{ \|u - U_j\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_j}^2(U_j, \mathcal{T}_j) \right\}^{-\frac{1}{2s}},$$

then we have

$$(7.70) \quad \begin{aligned} \|u - U_j\|_{H^1(\Omega)}^2 + \beta_2(\epsilon) \text{osc}_{\mathcal{T}_j}^2(U_j, \mathcal{T}_j) &\leq \|u - U_j\|_{H^1(\Omega)}^2 + \beta_2(\epsilon) \eta^2(U_j, \mathcal{T}_j) \\ &\leq (1 + C\beta_2(\epsilon)) (\|u - U_j\|_{H^1(\Omega)}^2 + \text{osc}_{\mathcal{T}_j}^2(U_j, \mathcal{T}_j)). \end{aligned}$$

Besides, by Theorem 54 and the Poincaré's inequality, we get

$$(7.71) \quad \|u - U_k\|_{H^1(\Omega)}^2 + \beta_2(\epsilon) \eta^2(U_k, \mathcal{T}_k) \leq \alpha^{k-j} \left\{ \|u - U_j\|_{H^1(\Omega)}^2 + \beta_2(\epsilon) \eta^2(U_j, \mathcal{T}_j) \right\}.$$

Combining (7.69)-(7.71), we have

$$(7.72) \quad \begin{aligned} \#\mathcal{T}_k - \#\mathcal{T}_0 &\leq C(1 + C\beta_2(\epsilon))^{\frac{1}{2s}} \left\{ \|u - U_k\|_{H^1(\Omega)}^2 + \beta_2(\epsilon) \eta^2(U_k, \mathcal{T}_k) \right\}^{-\frac{1}{2s}} \sum_{j=1}^k \alpha^{\frac{j}{2s}} \\ &\leq C |u, f|_s^{\frac{1}{s}} \left\{ \|u - U_k\|_{H^1(\Omega)}^2 + \beta_2(\epsilon) \eta^2(U_k, \mathcal{T}_k) \right\}^{-\frac{1}{2s}}. \end{aligned}$$

The theorem is proved. \square

