23

Exercises

23.1 Poisson introduction

1. Suppose that Ω is bounded and that $1 \leq p < q \leq \infty$. Prove that $L^q(\Omega) \subset L^p(\Omega)$. Is this inclusion still valid when Ω is not bounded? Give your justification.
2. Prove the convergence estimate of (23.1)

$$\mu_j - u(x_j) = O(h^2) \tag{23.1}$$

3. Find your weakest assumption on u so that the variational principle is valid. Prove your result.
4. Let $\Omega = (0,1)^2$ be partitioned by uniform grids of size h and consider the finite difference, finite element and finite volume methods for (23.7).

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{23.2}$$

where

$$\Delta = \begin{cases} \frac{d^2}{dx^2}, & d =, 1 \\ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, & d = 2. \end{cases}$$

   a) Prove that the stiffness matrices from all these three different methods are a scalar multiple of the matrix given (23.3).

$$A = tridiag(-I, B, -I), \quad B = tridiag(-1, 4, -1). \tag{23.3}$$

   b) After proper scaling, interpret the finite element and the finite volume methods as a finite difference method with right hand sides obtained by proper average of f.
   c) Derive the truncation errors for all the above three finite difference schemes.
5. Using any computing language, for linear finite element method for the Poisson equation with pure Dirichlet boundary condition,
   a) write a code for a uniform grid on the unit square and $(0,1)^2$ and solve the Poisson equation on the unit square with the exact solution given by

$$u(x, y) = \sin(x(1-x))y^2(1-y)^2. \tag{23.4}$$

   Assume that u_h is the linear finite element approximation defined on a uniform triangulation with mesh size h. Evaluate, for $h = 1/4, 1/8, 1/16$

$$\|u_I - u_h\|_{0,\Omega}, \quad |u_I - u_h|_{1,\Omega} \text{ and } \|u_I - u_h\|_{0,\infty,\Omega}$$

   where u_I is the interpolation of u. Determine the order of the above quantities with respect to h for each of the three different quadrature schemes (that are exact for $\mathcal{P}_k$ with $k = 0, 1, 2$ respectively) applied to evaluate the right hand side of the corresponding algebraic systems.

b) write a code for any general unstructured grid given by the data structures described in this chapter.

6. Consider the following two-point boundary value problem:

$$\begin{cases} Lu \equiv -\varepsilon u'' + u' = 0, & 0 < x < 1, \\ u(0) = 1, \ \ u(1) = 0, \end{cases} \tag{23.5}$$

the exact solution is

$$u(x) = \frac{1 - e^{-(1-x)/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

Design your finite difference, finite volume and finite element methods for solving the above problem for very small ε, say $\varepsilon = 10^{-8}$. Compare your numerical results with the exact solution for different mesh sizes.

7. Consider the following elliptic boundary value problems

$$\begin{aligned} -\nabla \cdot (a(x,y)\nabla)u &= f(x,y), \quad (x,y) \in \Omega, \\ \frac{\partial u}{\partial \nu} &= g(x,y), \quad (x,y) \in \partial\Omega \end{aligned}$$

where a is positive function and, f and g satisfy the consistent condition

$$\int_\Omega f = \int_{\partial\Omega} g \tag{23.6}$$

Assume that $\Omega = (0,1)^2$ which is partitioned by a uniform grid of size h.

a) Design a finite difference scheme and analyze its solvability
b) Design a finite element scheme and analyze its solvability
c) Design a finite volume scheme and analyze its solvability
d) Test and compare your results for a concrete problem that has the exact solution as given in (23.4)

8. Let $\Omega = (0,1)^2$ and consider a partition of Ω by a criss-cross grid (see Figure **??**) and its corresponding linear finite element space.

a) Using lexigraphical ordering of the grid, find the stiffness matrix for the corresponding finite element discretization of the Poisson equation.
b) At each grid point, derive the corresponding finite difference scheme and analyze the truncation error for such a finite difference scheme.

9. Consider the Poisson problem

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

Let f be piecewise constant on the initial mesh $\mathcal{T}_0$. Assume that $\mathcal{T}_{k+1}$ satisfies the interior node property with respect to $\mathcal{T}_k$, i.e. each element $\tau \in \mathcal{M}_k$, where $\mathcal{M}_k$ denotes the set of marked elements, contains at least one node of $\mathcal{T}_{k+1}$ in the interior of τ and each side of τ. An consequence of this property is the following discrete lower a posterior bound:

$$C_l \eta_k^2(u_k, \mathcal{M}_k) \le \|u_k - u_{k+1}\|_1^2,$$

where η_k is the posteriori estimator satisfies the following upper bound:

$$\|u - u_k\|_1^2 \le C_u \eta_k^2(u_k, \mathcal{T}_k).$$

Assume that Döruler marking strategy is used, i.e. the set $\mathcal{M}_K$ satisfies

$$\eta_k(u_k, \mathcal{M}_k) \ge \theta \eta_k(u_k, \mathcal{T}_k), \quad \theta \in (0,1].$$

Show that

$$\|u - u_{k+1}\|_1 \le \alpha \|u - u_k\|_1,$$

with some constant $\alpha \in (0,1]$.

10. (1)Derive the variational formulation for the Poisson equation with the Robin boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ (\alpha u + \beta \nabla u \cdot \mathbf{n}) = g, & \text{on } \partial\Omega, \end{cases} \tag{23.7}$$

(2) Write out the stiffness matrix of the finite element problem in the terms of basis functions.
(3) Derive the corresponding minimization problem, and prove the equivalence between the variational form and the minimization problem.

11. Let $e \in \mathcal{E}(\mathcal{T})$ be an interiori edge in the triangulation with nodes x_i and x_j, and shared by two triangles τ_1 and τ_2. Denoted the angle in τ opposing to e by θ_E^τ.

a) Derive the following identity

$$a_{ij} = -\frac{1}{2}(\cot\theta_E^{\tau_1} + \cot\theta_E^{\tau_2}).$$

b) Prove that $a_{ij} \leq 0$ if and only if the following Delaunay condition is satisfied:

$$\theta_E^{\tau_1} + \theta_E^{\tau_2} \leq \pi$$

c) † Prove that if the Delaunay condition is satisfied for all interior edges in the triangulation (such a triangulation is called Delaunay triangulation), the finite element solution is nonnegative for the equation $-\Delta u = f$ (with homogeneous Dirichlet boundary condition) if $f \geq 0$.

12. Consider a sequence of uniform grid on interval $[0, 1]$ as follows:

$$0 = x_0^k < x_1^k < \cdots < x_{N_k+1}^k = 1, \quad x_j^k = \frac{j}{N_k + 1}, \quad k = 1, 2, \cdots, J,$$

with $h_k = 1/(N_k + 1)$ and $N_k = 2^k - 1$. On each level k, we have the standard linear finite element space V_h^k with standard nodal basis functions ϕ_i^k. Denote all the interior nodes x_j^k, $j = 1, 2, \cdots, N_k$ on level k by $\mathcal{N}_k$, the so-called *hierarchical basis* (HB) refers to a special set of nodal basis functions

$$\{\phi_i^k : x_i^k \in \mathcal{N}_k \backslash \mathcal{N}_{k-1},\ k = 1, \cdots, J\}.$$

It is often to more convenient to use the scaled HB as follows

$$\{h_k\phi_i^k : x_i^k \in \mathcal{N}_k \backslash \mathcal{N}_{k-1},\ k = 1, \cdots, J\},$$

and with proper scaling, we shall denote the scaled HB by $\{\psi_i\}$, $i = 1, \cdots, 2^J - 1$ as shown in Figure 23.1.
If we use the HB to solve the Laplace problem

$$\begin{cases} -u'' = f, & \text{on } (0, 1), \\ u(0) = u(1) = 0. \end{cases}$$

on the J-th level uniform grid, show that the corresponding stiffness matrix is diagonal.

13. Define $\text{curl}\phi = (\frac{\partial\phi}{\partial x_2}, -\frac{\partial\phi}{\partial x_1})$ and $\text{rot}\mathbf{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}$, consider the following problem

$$\begin{aligned} -\text{graddiv}\mathbf{u} + \text{curlrot}\mathbf{u} &= \mathbf{f}, && \text{in } \Omega \\ \mathbf{n} \cdot \mathbf{u} &= g, && \text{on } \partial\Omega \\ \text{rot}\mathbf{u} &= h, && \text{on } \partial\Omega \end{aligned} \tag{23.8}$$

a) Prove that $-\text{graddiv} + \text{curlrot}$ is a vector Laplace, i.e.

$$-\text{graddiv} + \text{curlrot} = \begin{pmatrix} -\Delta & 0 \\ 0 & -\Delta \end{pmatrix}.$$

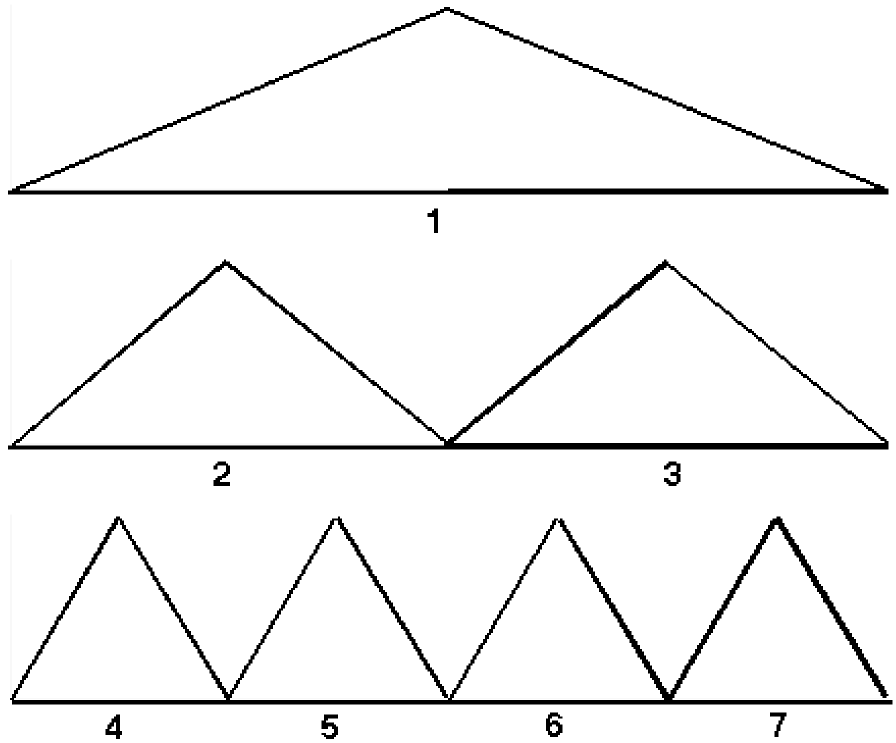

Fig. 23.1. One dimensional Hierarchical basis

b) Consider $g = 0$ and $h = 0$, and define the space

$$V := \{\mathbf{u} \in H^1(\Omega)^2 | \mathbf{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\}.$$

then the weak form of (23.8) is: Find $\mathbf{u} \in V$, such that

$$(\mathrm{rot}\mathbf{u}, \mathrm{rot}\mathbf{v}) + (\mathrm{div}\mathbf{u}, \mathrm{div}\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in V. \tag{23.9}$$

If we discretize the above weak form by finite element method on the unstructured grid, show that the stiffness matrix A is sparse and symmetric positive definite.

c) Consider the unit square domain $\Omega := [0, 1] \times [0, 1]$. Let the exact solution to be

$$\mathbf{u}_D = \begin{pmatrix} \cos(\pi x)\cos(\pi y) - 1 \\ \cos(\pi x)\cos(\pi y) - 1 \end{pmatrix},$$

and

$$\mathbf{f} = \begin{pmatrix} 2\pi^2 \cos(\pi x)\cos(\pi y) \\ 2\pi^2 \cos(\pi x)\cos(\pi y) \end{pmatrix}, \; g = \mathbf{u}_D \cdot \boldsymbol{n}, \text{ and } h = \mathrm{rot}\mathbf{u}_D,$$

- Use finite difference or finite element method to solve the problem (23.8) with the above given data $\mathbf{f}$, g, and h.
- Compare the numerical solution $\mathbf{u}_h$ with the given exact solution $\mathbf{u}_D$ and check whether it converges to the exact solution by computing $\frac{\|\mu_D-\mu_h\|_{l^2}}{\sqrt{N}}$ where μ_D and μ_h are the vectors corresponding to the exact solution $\mathbf{u}_D$ and $\mathbf{u}_h$ respectively, and N is the number of degrees of freedom.

d) Consider the L-shaped domain $\Omega := [-1,1]^2\backslash([0,1]\times[-1,0])$ as shown in Figure 23.2. Define

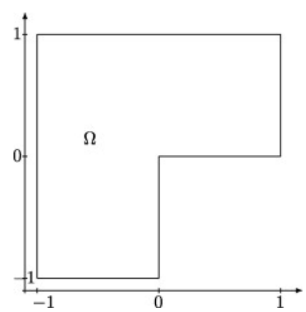

Fig. 23.2. L-shaped domain

$\phi := r^{2/3}\sin(2\theta/3)$ in polar coordinate and $\mathbf{u}_D := \text{curl}\phi$. Let

$$\mathbf{f} = \begin{pmatrix}0\\0\end{pmatrix},\ g = \mathbf{u}_D\cdot\boldsymbol{n},\ \text{and } h = \text{rot}\mathbf{u}_D,$$

- Verify that $\mathbf{u}_D$ is a solution to the problem (23.8). (Note that $\Delta\phi = 0$).
- Use finite difference or finite element method to solve the problem (23.8) with the above given data $\mathbf{f}$, g, and h.
- Compare the numerical solution with the given exact solution $\mathbf{u}_D$ and check whether it converges to the exact solution by computing $\frac{\|\mu_D-\mu_h\|_{l^2}}{\sqrt{N}}$ where μ_D and μ_h are the vectors corresponding to the exact solution $\mathbf{u}_D$ and $\mathbf{u}_h$ respectively, and N is the number of degrees of freedom.

14. Still consider the L-shaped domain, and use finite difference or finite element method to solve the problem (23.8) with

$$\mathbf{f} = \begin{pmatrix}1\\0\end{pmatrix},\ g = h = 0.$$

Compare the numerical solution with the following numerical solution, see Figure 23.3. If your numerical solution are different from the given numerical solution, which one is correct and explain your reasons.

23.2 Mesh

1. The triangulation $\{\mathcal{T}_h : h \in \aleph\}$ is said to be *shape-regular* if

$$\sup_{h\in\aleph}\max_{\tau\in\mathcal{T}_h}\frac{h_\tau}{\rho_\tau} \le \sigma_1 \qquad (23.10)$$

where ρ_τ denotes the radius of the ball inscribed in τ. Prove that (23.10) is equivalent to the following well-known minimum angle condition:

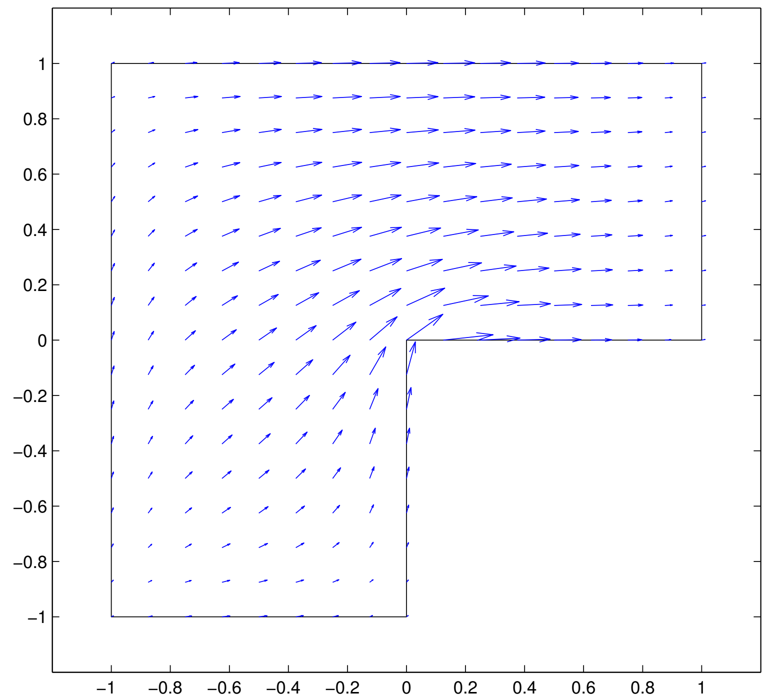

Fig. 23.3. Vector field of the numerical solution

$$\inf_{h\in\aleph}\min_{\tau\in\mathcal{T}_h}\theta_\tau \geq \theta_0 > 0 \tag{23.11}$$

where θ_τ is the minimum interior angle of τ for $\tau \in \mathcal{T}_h$ and θ_0 is a constant.

2. Prove that if the triangulation $\{\mathcal{T}_h : h \in \aleph\}$ satisfies (23.10), then

$$\sup_{h\in\aleph}\max_{\tau,\tau'\in\mathcal{T}_h,\ \tau\cap\tau'\neq\emptyset}\frac{h_\tau}{h_{\tau'}} \lesssim 1.$$

23.3 Linear finite element

1. Corresponding to the triangulations $\mathcal{T}_h$, the finite element spaces $\mathcal{S}^h = \mathcal{S}^h(\Omega)$ is defined by

$$\mathcal{S}^h(\Omega) = \{v \in C(\bar{\Omega}) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall\, \tau \in \mathcal{T}_h\}.$$

It is easy to see that $\mathcal{S}^h(\Omega) \subset H^1(\Omega)$. We further define that

$$\mathcal{S}_0^h(\Omega) = \mathcal{S}^h(\Omega) \cap H_0^1(\Omega).$$

a) Show that $\mathcal{S}^h$ is indeed a subspace of $H^1(\Omega)$.
b) Assume that $\partial\Omega$ is curved but piecewise convex. Define

$$\mathcal{S}^h = \{v : v \in S^h(\bar{\Omega}_h), v|_{\Omega\backslash\Omega_h} \text{ is the natural linear extention}\} \tag{23.12}$$

and

$$\mathcal{S}_0^h = \{v : v \in S_0^h(\bar{\Omega}_h), v|_{\Omega\backslash\Omega_h} = 0\}. \tag{23.13}$$

Prove that $\mathcal{S}^h \subset H^1(\Omega)$ and $\mathcal{S}_0^h \subset H_0^1(\Omega)$.
c) Prove, for the space $\mathcal{S}^h$ defined as above,

$$\|v\|_{0,\Omega\backslash\Omega_h} \lesssim h\|v\|_1 \quad \forall\, v \in \mathcal{S}^h.$$

2. a) Solve

$$\begin{cases} -\Delta u = -4, & (x,y) \in \Omega, \\ \quad u = x^2 + y^2, & (x,y) \in \partial\Omega, \end{cases}$$

on squared domain $[0,1] \times [0,1]$ with both structured mesh and unstructured mesh.
b) Solve

$$\begin{cases} -\Delta u = f(x,y), & (x,y) \in \Omega, \\ \quad u = x^2 + y^2, & (x,y) \in \Gamma_D, \\ \dfrac{\partial u}{\partial n} = 2x, & (x,y) \in \Gamma_N. \end{cases}$$

on the following square domain $[0,1] \times [0,1]$.

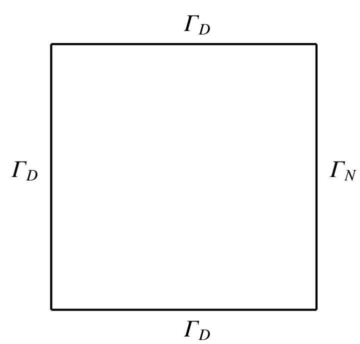

Fig. 23.4. *Square domain*

3. Given $g \in H^{-1/2}(\partial\Omega)$, define a proper weak solution to the following boundary value problem

$$\begin{cases} \Delta u = 0, \ \ in \ \Omega, \\ u = g, \ \ on \ \partial\Omega. \end{cases}$$

If $u_h \in S^h$ is the discrete harmonic function such that $u_h = Q_h g$ on $\partial\Omega$, prove that

$$\|u - u_h\| \leq h^s \|g\|_{-\frac{1}{2}+s,\partial\Omega}, \ 0 \leq s \leq 1.$$

4. Solve

$$\begin{cases} -\Delta u = -2e^{x+y}, & (x,y) \in \Omega, \\ u = e^{x+y}, & (x,y) \in \Gamma_D, \\ \dfrac{\partial u}{\partial n} = e^{x+y}, & (x,y) \in \Gamma_N. \end{cases}$$

using linear finite element on the following domain with structured triangular mesh. And then verify numerically that

$$\|u - u_h\|_0 + h\|u - u_h\|_1 = O(h^2).$$

(Hint: Try to use the data structures introduced in class.)

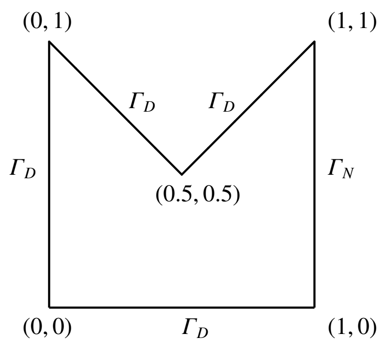

Fig. 23.5. *Computational domain*

5. Let T be a triangle element with vertices x_i and λ_i are the corresponding barycentric coordinates, $i = 1, 2, 3$. Let $\mathbf{n}_i$ be the unit outward normal vector of the face F_i and d_i be the distance from x_i to F_i. Prove that

$$\nabla \lambda_i = -\frac{1}{d_i}\mathbf{n}_i.$$

6. Let $\Omega \in \mathbb{R}^2$ be a simply connected domain. Define the Sobolev space:

$$H(\mathrm{div}, \Omega) := \{\mathbf{v} \in (L^2(\Omega))^2 : \mathrm{div}\mathbf{v} \in L^2(\Omega)\},$$

where $\mathrm{div} := (\partial_x, \partial_y)^\top$. Define also the homogeneous space

$$H_0(\mathrm{div}, \Omega) := \{\mathbf{v} \in H(\mathrm{div}, \Omega) : \mathbf{v} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\}.$$

The exact sequence below is well known.

$$H_0^1(\Omega) \xrightarrow{\mathbf{curl}} H_0(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega). \tag{23.14}$$

Here the exact sequence (23.25) means:

$$\{\mathbf{v} \in H_0(\mathrm{div}, \Omega) : \mathrm{div}\mathbf{v} = 0\} = \mathrm{curl} H_0^1(\Omega), \text{ and } L_0^2(\Omega) = \mathrm{div} H_0(\mathrm{div}, \Omega). \tag{23.15}$$

a) Define the Raviart-Thomas elements by $FEM_{\mathrm{div}} := (K^d, P_K^d, N_K^d)$, where
   i. K^d is a triangule;
   ii. $P_K^d = \underset{\sim}{\alpha} + \beta \begin{bmatrix} x \\ y \end{bmatrix}$, where $\underset{\sim}{\alpha} \in \mathbb{R}^2$ is a constant vector and $\beta \in \mathbb{R}$.

iii. $N_K^d = \{d_{i,K}^d\}_{i=1,2,3}$, with $d_{i,K}^d(\underset{\sim}{\varphi}) = \int_{e_i} \underset{\sim}{\varphi} \cdot \mathbf{n}_i ds$, where e_i is the edge of K, and $\mathbf{n}_i$ the normal unit vector of e_i.

Prove that the finite element is unisolvent, and give an explicit formulation of their nodal basis functions using barycentric coordinates.

b) Given a triangular triangulation $\mathcal{T}_h$, define the finite element spaces

$$H_h^{RT}(\mathrm{div}, \Omega) := \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_h \in P_K^d \ \forall K \in \mathcal{T}_h, \ \int_e \mathbf{v}_h \cdot \mathbf{n}_e \text{ is continuous across } e\},$$

and

$$H_{h0}^{RT}(\mathrm{div}, \Omega) := \{\mathbf{v}_h \in H^{RT}(\mathrm{div}, \Omega) : \int_e \mathbf{v}_h \cdot \mathbf{n}_e = 0 \text{ on } e \subset \partial\Omega\}.$$

Prove that $H_h^{RT}(\mathrm{div}, \Omega) \subset H(\mathrm{div}, \Omega)$ and $H_{h0}^{RT}(\mathrm{div}, \Omega) \subset H_0(\mathrm{div}, \Omega)$.

c) Define the nodal interpolation operators by

$$I_h^d : H(\mathrm{div}, \Omega) \to H_h^{RT}(\mathrm{div}, \Omega), \text{ such that } \int_e I_h^d \mathbf{v} \cdot \mathbf{n}_e = \int_e \mathbf{v} \cdot \mathbf{n}_e.$$

Show that

$$I_h^d H_0(\mathrm{div}, \Omega) = H_{h0}^{RT}(\mathrm{div}, \Omega).$$

d) Let Q_h be the space of piecewise constants defined on $\mathcal{T}_h$, and $\mathring{Q}_h = Q_h \cap L_0^2(\Omega)$, where $L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}$. Moreover, let $\Pi_0 : L^2(\Omega) \to Q_h$ be the projection operator. Prove the commutativity hold below:

$$\Pi_0 \circ \mathrm{div} = \mathrm{div} \circ I_h^d, \quad \text{on } H(\mathrm{div}, \Omega).$$

e) Let V_h be the continuous linear element on $\mathcal{T}_h$, and $V_{h0} = V_h \cap H_0^1(\Omega)$. Prove the discrete exact sequences below:

$$V_{h0} \xrightarrow{\mathbf{curl}} H_{h0}^{RT}(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} \mathring{Q}_h, \tag{23.16}$$

The meaning of the discrete exact sequences are similar to (23.15), namely

$$\{\mathbf{v} \in H_{h0}^{RT}(\mathrm{div}, \Omega) : \mathrm{div}\mathbf{v} = 0\} = \mathrm{curl} V_{h0}(\Omega), \quad \text{and } \mathring{Q}_h = \mathrm{div} H_{h0}(\mathrm{rot}, \Omega).$$

23.4 Sobolev Spaces

1. The Sobolev space of index (m, p) is defined by

$$W^{m,p}(\Omega) \overset{\text{def}}{=} \{v \in L^p(\Omega) : D^\alpha v \in L^p(\Omega) \ \text{ if } |\alpha| \le m\},$$

with a norm $\|\cdot\|_{m,p,\Omega}$ given by

$$\|v\|_{m,p,\Omega}^p \overset{\text{def}}{=} \sum_{|\alpha| \le m} \|D^\alpha v\|_{L^p(\Omega)}^p, \tag{23.17}$$

We will have occasions to use the seminorm $|\cdot|_{m,p,\Omega}$ given by

$$|v|_{m,p,\Omega}^p \overset{\text{def}}{=} \sum_{|\alpha| = m} \|D^\alpha v\|_{L^p(\Omega)}^p.$$

Prove that $W^{m,p}(\Omega)$ is a Banach space.

2. Given $s = m + \sigma$ with $\sigma \in (0,1)$ and integer $m \geq 0$, define

$$W^{s,p}(\Omega) = \{v \in W^{m,p}(\Omega) : \ |v|_{s,p} < \infty\}$$

where

$$|v|^p_{s,p,\Omega} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha v(x) - D^\alpha v(y)|^p}{|x-y|^{n+\sigma p}} dx\, dy \tag{23.18}$$

with a norm $\|\cdot\|_{s,p}$ given by

$$\|v\|^p_{s,p,\Omega} = \|v\|^p_{m,p} + |v|^p_{s,p}. \tag{23.19}$$

Prove that $W^{s,p}(\Omega)$ is a Banach space.

3. Assume that $\Omega = \Omega_1 \cup \Omega_2$ with $d = \text{dist}\,(\Omega_1 \setminus \Omega_2, \Omega_2 \setminus \Omega_1) > 0$. Then

$$\|v\|_{s,p,\Omega} \lesssim d^{-(s+\frac{n}{p})}(\|v\|_{s,p,\Omega_1} + \|v\|_{s,p,\Omega_2})$$

4. Prove

$$(1 + |\xi|^2)^m \leq \sum_{|\alpha|\leq m} \xi^{2\alpha} \lesssim (1 + |\xi|^2)^m. \tag{23.20}$$

5. Prove

$$\int_{\mathbb{R}^n} \frac{|e^{i\xi\cdot\eta} - 1|^2}{|\eta|^{n+2s}} d\eta = C(n,s)|\xi|^{2s}, \quad 0 < s < 1 \tag{23.21}$$

where $C(n,s)$ is a positive constant depending on n and s.

6. Assume that a function v is defined in Ω, and $\widetilde{v}$ is the zero extension of v to $\mathbb{R}^n$, i.e. $\widetilde{v}$ equals to v in Ω, and equals to zero in $\mathbb{R}^n$
Omega. Define

$$\widetilde{W}^{s,p}_0(\Omega) = \{v \in W^{s,p}(\Omega) : \ \widetilde{v} \in W^{s,p}(\mathbb{R}^n)\},$$

with a norm

$$\|v\|_{\widetilde{s,p,\Omega}} := \|\widetilde{v}\|_{s,p,\mathbb{R}^n}.$$

Prove that $\widetilde{W}^{s,p}_0(\Omega)$ is a Banach space.

7. Prove that, if $mp > d$, then

$$\|uv\|_{m,p,\Omega} \lesssim \|u\|_{m,p,\Omega}\|v\|_{m,p,\Omega}.$$

8. For $\sigma \in (0,1)$, prove that

$$\int_{\Omega^c} \frac{dy}{|x-y|^{n+\sigma p}} \cong \frac{1}{\text{dist}(x,\partial\Omega)^{\sigma p}}.$$

9. Given $s > 0$, the Sobolev space $H^{-s}(\Omega)$ is defined to be the dual space of $H^s(\Omega)$, namely

$$H^{-s}(\Omega) = \left(H^s_0(\Omega)\right)^*.$$

Prove that the space $H^{-s}(\Omega)$ defined as above is identical to the space that is the completion of the space $L^2(\Omega)$ iwth respect to the following Lax negative norm:

$$\|v\|_{-s,\Omega} = \sup_{\phi\in H^s(\Omega)} \frac{(v,\phi)_{0,\Omega}}{\|\phi\|_{s,\Omega}}.$$

Furthermore, the above norm is equivalent to the dual norm for $H^{-s}(\Omega)$.

10. Let B_0 and B_1 be two Banach spaces with B_1 continuously embedded and dense in B_0, definite for $p \in [1, \infty)$

$$K(t,u) = \inf_{u_0+u_1=u} \left(\|u_0\|_{H_0}^p + t^p \|u_1\|_{H_1}^p \right)^{1/p}, \; u_0 \in B_0, \; u_1 \in B_1.$$

The intermediate space $B_{s,p}$ is defined by

$$B_{s,p} = \left\{ u \in B_0 : \|u\|_{B_{s,p}} < \infty \right\},$$

where

$$\|u\|_{B_{s,p}} = \left(\int_0^\infty t^{-sp} K^p(t,u) \frac{dt}{t} \right)^{1/p}.$$

Prove that for any $u \in B_1$,

$$\|u\|_{B_{s,p}} \leq C_s \|u\|_{B_0}^{1-s} \|u\|_{B_1}^s,$$

with $C_s = (ps(1-s))^{-1/p}$.

23.5 Interpolation, Projection and Error Estimates

1. Explain directly why L^2 projection can be extended to the space H^{-1}.
2. Define Galerkin projection $P_h : H_0^1(\Omega) \to S_0^h$ as

$$A(P_h u, v) = A(u, v), \; \forall u \in H_0^1(\Omega), \; v \in S_0^h.$$

Prove that $\lim_{h \to 0} P_h u = u$, for any $u \in H_0^1(\Omega)$.

3. P_h is the Galerkin projection defined as above. Assume that $U \in H_0^1(\Omega)$ is the solution to

$$A(U, \chi) = (F, \chi), \; \forall \chi \in H_0^1(\Omega).$$

Prove that

$$\|U - P_h U\| \geq C \|U - P_h U\|_1^2 / \|F\|.$$

4. P_h is the Galerkin projection defined as above. Construct a counter example to show that P_h cannot be stable in L^p norm if $p < \infty$.
5. Establish the following estimate

$$\|u - P_h u\|_{1-\alpha} \lesssim h^{\alpha - \frac{1}{2} + \varepsilon} \|u\|_{\frac{1}{2}+\varepsilon}, \; 0 < \varepsilon \leq 3/2.$$

6. Define L^2 projection $Q_h : L^2(\Omega) \to S_0^h$ as

$$(Q_h u, v) = (u, v), \; \forall u \in L^2(\Omega), \; v \in S_0^h.$$

23.6 Discontinuous Galerkin

1. Consider the following two-point boundary value problem:

$$\begin{cases} Lu \equiv -\varepsilon u'' + u' = 0, & 0 < x < 1, \\ u(0) = 1, \ \ u(1) = 0, \end{cases} \tag{23.22}$$

the exact solution is

$$u(x) = \frac{1 - e^{-(1-x)/\varepsilon}}{1 - e^{-1/\varepsilon}}.$$

Design your finite difference, finite volume and finite element methods for solving the above problem for very small ε, say $\varepsilon = 10^{-8}$. Compare your numerical results with the exact solution for different mesh sizes.

2. Consider the quadratic finite element on a triangular in 2-D and construct its nodal basis $\{\phi_i\}_{i=1}^6$ by barycenter coordinates such that

$$\phi_i(x_j) = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

where x_1, x_2, and x_3 are the vertices of the triangular and x_4, x_5, and x_6 are the midpoints of the edges.

3. We consider the 3-D Morley element, which is defined by the triple (T, P_T, D_T) with
   - T is a tetrahedron;
   - $P_T = P_2(T)$;
   - $D_T = \{d_i\}_{i=1:10}$, such that for $\varphi \in H^2(T)$,

$$d_i(\varphi) = \fint_{F_i} \frac{\partial \varphi}{\partial \mathbf{n}_{F_i}}, \ F_i \text{ is a face of } T, \ i = 1:4$$

   and

$$d_{i+4}(\varphi) = \fint_{e_i} \varphi, \ e_i \text{ is an edge of } T, \ i = 1:6.$$

See Figure 23.6(left). The finite element space is defined by

$$M_h = \{w_h \in L^2(\Omega): \ w_h|_T \in P_2(T), \ \fint_e w_h \text{ is continuous at any edge } e,$$
$$\text{and } \fint_F \frac{\partial w_h}{\partial_{\mathbf{n}_F}} \text{ is continuous across any interior face } F\}.$$

   a) Prove that the 3-D Morley element is unisolvent.
   b) Let F be an interior face of the triangulation. Prove that $\fint_F \nabla w_h$ is continuous across the face for any $w_h \in M_h$.

4. Let M_{h0} be the Morley element space associated with $H_0^2(\Omega)$, and V_{h0}^{CR} be the Crouzeix-Raviart element space associated with $H_0^1(\Omega)$. Denote $\mathbf{V}_{h0}^{\text{CR}} := V_{h0}^{\text{CR}} \times V_{h0}^{\text{CR}}$. Let ∇_h be the piecewise-defined gradient operator; let $\mathbf{curl}_h$ be the rotation of ∇_h; and let div_h be the piecewise-defined divergence operator. Define $\widetilde{\mathbf{V}}_{h0}^{\text{CR}} := \{\mathbf{v}_h \in \mathbf{V}_{h0}^{\text{CR}} : \text{div}_h \mathbf{v}_h = 0\}$. Show that $\mathbf{curl}_h M_{h0} = \widetilde{\mathbf{V}}_{h0}^{\text{CR}}$.
5. The rectangle Morley element is defined by (Q, P_Q^M, D_Q^M) with
   a) Q is a rectangle;
   b) $P_Q^M = P_2(Q) + span\{x^3, y^3\}$;

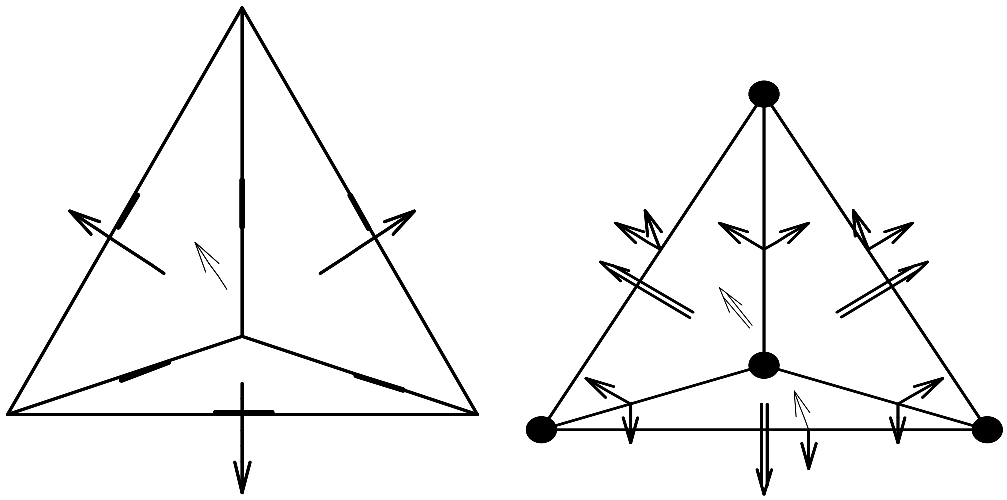

Fig. 23.6. Illustration of the 3D elements. Left: 3D MWX element; right: WX element for 3D 6th order problem.

c) the components of $D_Q^M = \{d_i^M, d_{i+4}^M\}_{i=1:4}$ for any $v \in H^2(Q)$ are:

$$d_i^M(v) = v(a_i),\ a_i \text{ the vertices of } Q;$$

$$d_{i+4}^M(v) = \fint_{e_i} \partial_{\mathbf{n}_{e_i}} v ds,\ e_i \text{ the edges of } Q.$$

a) Prove the P_Q unisolvence of the rectangle Morley element;
b) Establish a set of nodal basis functions.

6. We consider the 3-D WX element for 6th order problem, which is defined by the triple (T, P_T, D_T) with
- T is a tetrahedron;
- $P_T = P_3(T)$;
- $D_T = \{d_i\}_{i=1:20}$, such that for $\varphi \in H^3(T)$

$$d_i(\varphi) = \fint_{F_i} \frac{\partial^2 \varphi}{\partial \mathbf{n}_{F_i}^2},\ F_i \text{ is the face of } T,\ i = 1:4.$$

$$d_{2i+3}(\varphi) = \fint_{e_i} \frac{\partial \varphi}{\partial \nu_{e_i}^L},\ d_{2i+4}(\varphi) = \fint_{e_i} \frac{\partial \varphi}{\partial \nu_{e_i}^R},\ e_i \text{is the edge of } T,$$
$$\nu_{e_i}^L \text{ and } \nu_{e_i}^L \text{ is the two unit vectors perpendicular to } e_i, i = 1:6.$$

and

$$d_{i+16}(\varphi) = \varphi(a_i),\ a_i \text{ is the vertices of } T,\ i = 1:4.$$

See Figure 23.6(right). The finite element space is defined by

$$WX_h = \{w_h \in L^2(\Omega):\ w_h|_T \in P_3(T),\ \fint_F \frac{\partial^2 w_h}{\partial \mathbf{n}_F^2} \text{ is continuous across interior face } F,$$
$$\fint_e \frac{\partial w_h}{\partial \nu_e^L} \text{ and } \fint_e \frac{\partial w_h}{\partial \nu_e^R} \text{ are continuous at edge } e, \text{ and } w_h(a) \text{ is continuous at vertex } a\}.$$

a) Prove that the 3-D WX element for 6th order problem is unisolvent.
b) Let F be an interior face of the triangulation. Prove that $\fint_F \nabla^2 w_h$ is continuous across the face for any $w_h \in M_h$.
c) Let e be an edge of the triangulation. Prove that $\fint_e \nabla w_h$ is continuous at the edge for any $w_h \in M_h$.

7. Consider the Kirchhoff plate model

$$\min_{w \in H^2(\Omega) \cap H_0^1(\Omega)} \frac{1}{2}\left[(1-\nu)\|\nabla^2 w\|_{0,\Omega}^2 + \nu\|\Delta w\|_{0,\Omega}^2\right] - (f, w)$$

The minimizer $u \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfies the following 4-th order PDE

$$\begin{cases} \Delta^2 u = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ Bu = 0, & \text{on } \partial\Omega. \end{cases}$$

Derive the explicit form of the operator B.

8. Consider the Poisson equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{23.23}$$

Discretized by the non-conforming P1 finite element method ((5.2.1) in the lecture note). Describe and prove its consistency and stability. (Please use the definitions of consistency and stability in the lecture notes (Chapter 7) to prove your results)

9. For the Poisson equation (23.23) discretized by the following non-symmetric interior discontinuous Galerkin (NIPG) method: Find $u_h \in V_h = \{v_h \in L^2(\Omega) : \ v_h|_K \in P_1(K) \ \forall\, K \in \mathcal{T}_h\}$

$$B_h(u_h, v_h) = \int_\Omega f v_h dx, \quad \forall v_h \in V_h,$$

where

$$B_h(u_h, v_h) := \int_\Omega \nabla_h u_h \cdot \nabla_h v_h \, dx - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] \, ds + \int_{\mathcal{E}_h} [u_h] \cdot \{\nabla_h v_h\} \, ds + \int_{\mathcal{E}_h} \frac{\eta}{h_e} [u_h] \cdot [v_h] \, ds,$$

Describe and prove its consistency and stability. (Please use the definitions of consistency and stability in the lecture notes (Chapter 7) to prove your results)

23.7 Mixed finite element

1. Find conditions for $A \in R^{n\times n}$ and $B \in R^{n\times m}$ such that the following matrix is nonsingular:

$$\begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix}$$

2. Assuem that $A \in R^{n\times n}$ is SPD and $B \in R^{n\times m}$ is of full-rank. Prove that the following "preconditioned matrix" (with $S = BA^{-1}B^*$):

$$\begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}^{-1} \begin{pmatrix} A & B^* \\ B & 0 \end{pmatrix}$$

only has three distinctive eigenvalues: $1, (1 \pm \sqrt{5})/2$. Determine the multiplicity of these eigenvalues.

3. Let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times n}$, such that A is symmetric positive definite on $ker(B)$. Show that

$$\min_{Bx=0}(\frac{1}{2}x^t Ax - bx)$$

exists uniquely.

4. We consider the computational domain $[0,1]\times[0,1]$ with the uniform grid as shown in Figure 23.16. We use the $P_2^0 - P_0^{-1}$ element to discretize the Stokes problem

$$\begin{cases} -\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \\ \text{div}\mathbf{u} = 0 \text{ in } \Omega, \\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega. \end{cases} \tag{23.24}$$

   a) Write out the finite element problem;
   b) Write out the stiffness matrix of the finite element problem in the form of $\begin{bmatrix} A & B^t \\ B & 0 \end{bmatrix}$, and write A and B in terms of basis functions.

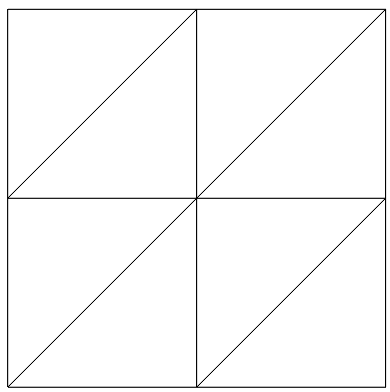

Fig. 23.7. The computational domain.

5. Prove that $\det(A)$ is the unique function of n variables satisfying that
   - it is multilinear,
   - it is skewsymmetric,
   - $\det(e_1, e_2, \cdots, e_n) = 1$.

   Hint: Denote the i-th column of A by A_i. We consider $\det(A)$ as $\det(A_1, A_2, \cdot, A_n)$. By being multilinear, it means that $\det(\alpha_1 A_1, \cdots, \alpha_n A_n) = \alpha_1\cdots\alpha_n \det(A_1,\cdots,A_n)$, and $\det(A_1,\ldots,A_i^{(1)}+A_i^{(2)},\ldots,A_n) = \det(A_1,\ldots,A_i^{(1)},\ldots,A_n) + \det(A_1,\ldots,A_i^{(2)},\ldots,A_n)$, for any i and any $A_i^{(1)}$ and $A_i^{(2)}$. By being skewsymmetric, it means $\det(\cdots, A_i, \cdots, A_j \cdots) = -\det(\cdots, A_j, \cdots, A_i \cdots)$, $\forall i \neq j$.
6. Show that exterior derivatives satisfy $d_{i+1}d_i = 0$, $0 \le i \le n-1$. Consider $n = 2$ and 3.
7. Let K be a tetrahedron, and f be a face of K. Denote $\mathbf{u} = \mathbf{a} + \mathbf{b}\times\mathbf{x}$. Show that if $\int_e \mathbf{u}\cdot\tau_e = 0$ for any edge e of f, then $\mathbf{u}\times\mathbf{n}_f = 0$, with $\mathbf{n}_f$ the normal vector of f.
8. Let $\Omega \in \mathbb{R}^2$ be a simply connected domain. Define the Sobolev space:

$$H(\text{div}, \Omega) := \{\mathbf{v} \in (L^2(\Omega))^2 : \text{div}\mathbf{v} \in L^2(\Omega)\},$$

   where div $:= (\partial_x, \partial_y)^\top$. Define also the homogeneous space

$$H_0(\mathrm{div},\Omega) := \{\mathbf{v} \in H(\mathrm{div},\Omega) : \mathbf{v}\cdot\mathbf{n} = 0, \text{ on } \partial\Omega\}.$$

The exact sequence below is well known.

$$H_0^1(\Omega) \xrightarrow{\mathbf{curl}} H_0(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} L_0^2(\Omega). \tag{23.25}$$

Here the exact sequence (23.25) means:

$$\{\mathbf{v} \in H_0(\mathrm{div},\Omega) : \mathrm{div}\mathbf{v} = 0\} = \mathrm{curl}H_0^1(\Omega), \quad \text{and } L_0^2(\Omega) = \mathrm{div}H_0(\mathrm{div},\Omega).$$

a) Define the Raviart-Thomas elements by $FEM_{\mathrm{div}} := (K^d, P_K^d, N_K^d)$, where
   i. K^d is a triangule;
   ii. $P_K^d = \underset{\sim}{\alpha} + \beta\begin{bmatrix} x \\ y \end{bmatrix}$, where $\underset{\sim}{\alpha} \in \mathbb{R}^2$ is a constant vector and $\beta \in \mathbb{R}$.
   iii. $N_K^d = \{d_{i,K}^d\}_{i=1,2,3}$, with $d_{i,K}^d(\underset{\sim}{\varphi}) = \int_{e_i} \underset{\sim}{\varphi}\cdot\mathbf{n}_i ds$, where e_i is the edge of K, and $\mathbf{n}_i$ the normal unit vector of e_i.

   Prove that the finite element is unisolvent, and give an explicit formulation of their nodal basis functions using barycentric coordinates.
b) Given a triangular triangulation $\mathcal{T}_h$, define the finite element spaces

$$H_h^{RT}(\mathrm{div},\Omega) := \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_h \in P_K^d\ \forall\, K \in \mathcal{T}_h, \int_e \mathbf{v}_h\cdot\mathbf{n}_e \text{ is continuous across } e\},$$

   and

$$H_{h0}^{RT}(\mathrm{div},\Omega) := \{\mathbf{v}_h \in H^{RT}(\mathrm{div},\Omega) : \int_e \mathbf{v}_h\cdot\mathbf{n}_e = 0 \text{ on } e \subset \partial\Omega\}.$$

   Prove that $H_h^{RT}(\mathrm{div},\Omega) \subset H(\mathrm{div},\Omega)$ and $H_{h0}^{RT}(\mathrm{div},\Omega) \subset H_0(\mathrm{div},\Omega)$.
c) Define the nodal interpolation operators by

$$I_h^d : H(\mathrm{div},\Omega) \to H_h^{RT}(\mathrm{div},\Omega), \text{ such that } \int_e I_h^d\mathbf{v}\cdot\mathbf{n}_e = \int_e \mathbf{v}\cdot\mathbf{n}_e.$$

   Show that

$$I_h^d H_0(\mathrm{div},\Omega) = H_{h0}^{RT}(\mathrm{div},\Omega).$$

d) Let Q_h be the space of piecewise constants defined on $\mathcal{T}_h$, and $\mathring{Q}_h = Q_h \cap L_0^2(\Omega)$, where $L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}$. Moreover, let $\Pi_0 : L^2(\Omega) \to Q_h$ be the projection operator. Prove the commutativity hold below:

$$\Pi_0 \circ \mathrm{div} = \mathrm{div} \circ I_h^d, \quad \text{on } H(\mathrm{div},\Omega).$$

e) Let V_h be the continuous linear element on $\mathcal{T}_h$, and $V_{h0} = V_h \cap H_0^1(\Omega)$. Prove the discrete exact sequences below:

$$V_{h0} \xrightarrow{\mathbf{curl}} H_{h0}^{RT}(\mathrm{div},\Omega) \xrightarrow{\mathrm{div}} \mathring{Q}_h, \tag{23.26}$$

   The meaning of the discrete exact sequences are similar to that of (23.27) and (23.25). Namely

$$\{\mathbf{v} \in H_{h0}^{RT}(\mathrm{div},\Omega) : \mathrm{div}\mathbf{v} = 0\} = \mathrm{curl}V_{h0}(\Omega), \quad \text{and } \mathring{Q}_h = \mathrm{div}H_{h0}(\mathrm{rot},\Omega).$$

9. Let $\Omega \in \mathbb{R}^2$ be a simply connected domain. Define the Sobolev spaces:

$$H(\mathrm{rot},\Omega) := \{\mathbf{v} \in (L^2(\Omega))^2 : \mathrm{rot}\mathbf{v} \in L^2(\Omega)\},$$

where $\mathrm{rot} := (-\partial_y, \partial_x)^\top$. Define also the homogeneous space

$$H_0(\mathrm{rot},\Omega) := \{\mathbf{v} \in H(\mathrm{rot},\Omega) : \mathbf{v} \times \mathbf{n} = 0, \text{ on } \partial\Omega\}.$$

The exact sequence below is well known.

$$H_0^1(\Omega) \xrightarrow{\mathbf{grad}} H_0(\mathrm{rot},\Omega) \xrightarrow{\mathrm{rot}} L_0^2(\Omega). \tag{23.27}$$

Here the exact sequence means:

$$\{\mathbf{v} \in H_0(\mathrm{rot},\Omega) : \mathrm{rot}\mathbf{v} = 0\} = \mathrm{grad}H_0^1(\Omega), \text{ and } L_0^2(\Omega) = \mathrm{rot}H_0(\mathrm{rot},\Omega).$$

a) Define $FEM_{\mathrm{rot}} := (K^r, P_K^r, N_K^r)$, where
   i. K^r is a triangule;
   ii. $P_K^r = \underset{\sim}{\alpha} + \beta \begin{bmatrix} -y \\ x \end{bmatrix}$, where $\underset{\sim}{\alpha} \in \mathbb{R}^2$ is a constant vector and $\beta \in \mathbb{R}$.
   iii. $N_K^r = \{d_{i,K}^r\}_{i=1,2,3}$, with $d_{i,K}^r(\underset{\sim}{\varphi}) = \int_{e_i} \underset{\sim}{\varphi} \cdot \tau_{e_i} ds$, where e_i is the edge of K, and τ_{e_i} the tangential unit vector of e_i.

   Prove that the finite element is unisolvent, and give an explicit formulation of their nodal basis functions using barycentric coordinates.
b) Again, given a triangular triangulation $\mathcal{T}_h$, define the finite element spaces

$$H_h^{RT}(\mathrm{rot},\Omega) := \{\mathbf{v}_h \in (L^2(\Omega))^2 : \mathbf{v}_h \in P_K^r \ \forall K \in \mathcal{T}_h, \int_e \mathbf{v}_h \cdot \tau_e \text{ is continuous across } e\}.$$

   and

$$H_{h0}^{RT}(\mathrm{rot},\Omega) := \{\mathbf{v}_h \in H^{RT}(\mathrm{rot},\Omega) : \int_e \mathbf{v}_h \cdot \tau_e = 0 \text{ on } e \subset \partial\Omega\}.$$

   Prove that $H_h^{RT}(\mathrm{rot},\Omega) \subset H(\mathrm{rot},\Omega)$ and $H_{h0}^{RT}(\mathrm{rot},\Omega) \subset H_0(\mathrm{rot},\Omega)$.
c) Define the nodal interpolation operators by

$$I_h^r : H(\mathrm{rot},\Omega) \to H_h^{RT}(\mathrm{rot},\Omega), \text{ such that } \int_e I_h^r\mathbf{v} \cdot \tau_e = \int_e \mathbf{v} \cdot \tau_e.$$

   Show that

$$I_h^r H_0(\mathrm{rot},\Omega) = H_{h0}^{RT}(\mathrm{rot},\Omega).$$

d) Let Q_h be the space of piecewise constants defined on $\mathcal{T}_h$, and $\mathring{Q}_h = Q_h \cap L_0^2(\Omega)$, where $L_0^2(\Omega) = \{q \in L^2(\Omega), \int_\Omega q = 0\}$. Moreover, let $\Pi_0 : L^2(\Omega) \to Q_h$ be the projection operator. Prove the commutativity hold below:

$$\Pi_0 \circ \mathrm{rot} = \mathrm{rot} \circ I_h^r, \text{ on } H(\mathrm{rot},\Omega),$$

e) Let V_h be the continuous linear element on $\mathcal{T}_h$, and $V_{h0} = V_h \cap H_0^1(\Omega)$. Prove the discrete exact sequences below:

$$V_{h0} \xrightarrow{\mathbf{grad}} H_{h0}^{RT}(\mathrm{rot},\Omega) \xrightarrow{\mathrm{rot}} \mathring{Q}_h. \tag{23.28}$$

   The meaning of the discrete exact sequences is

$$\{\mathbf{v} \in H_{h0}^{RT}(\mathrm{rot},\Omega) : \mathrm{rot}\mathbf{v} = 0\} = \mathrm{grad}V_{h0}(\Omega), \text{ and } \mathring{Q}_h = \mathrm{rot}H_{h0}(\mathrm{rot},\Omega).$$

10. Let U and V be two Hilbert spaces, with inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$ respectively. Let $\mathcal{B}(\cdot,\cdot) : U\times V \to \mathbb{R}$ be a continuous bilinear form

$$\mathcal{B}(u,v) \le \|\mathcal{B}\|\|u\|_U\|v\|_V. \tag{23.29}$$

Assume the inf-sup condition holds:

$$\inf_{u\in U}\sup_{v\in V}\frac{\mathcal{B}(u,v)}{\|u\|_U\|v\|_V} = \inf_{v\in V}\sup_{u\in U}\frac{\mathcal{B}(u,v)}{\|u\|_U\|v\|_V} = \alpha > 0. \tag{23.30}$$

Let $U_h \subset U$ and $V_h \subset V$ be two nontrivial subspaces of U and V respectively, and the discrete inf-sup condition holds:

$$\inf_{u_h\in U_h}\sup_{v_h\in V_h}\frac{\mathcal{B}(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} = \inf_{v_h\in V_h}\sup_{u_h\in U_h}\frac{\mathcal{B}(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} = \alpha_h > 0. \tag{23.31}$$

Consider the following variational problem: Find $u \in U$ such that

$$\mathcal{B}(u,v) = \langle f, v\rangle,\ \forall\, v \in V, \tag{23.32}$$

where $f \in V^*$ and $\langle\cdot,\cdot\rangle$ is the pairing between V^* and V, and its discretisation: find $u_h \in U_h$, such that

$$\mathcal{B}(u_h,v_h) = \langle f, v_h\rangle,\ \forall\, v_h \in V_h. \tag{23.33}$$

Prove

$$\|u-u_h\|_U \le \frac{\|\mathcal{B}\|}{\alpha_h}\inf_{w_h\in U_h}\|u-w_h\|_U. \tag{23.34}$$

11. Let $\Omega \subset \mathbb{R}^2$ be a connected polygonal domain, and $\mathcal{T}_h$ be a triangle triangulation of Ω. Let $V_{h0}^{\rm CR}$ be the nonconforming P_1 element space associated with $H_0^1(\Omega)$, and denote $\mathbf{V}_{h0}^{\rm CR} := V_{h0}^{\rm CR}\times V_{h0}^{\rm CR}$. Define $\tilde{Q}_h := \{q_h \in L^2(\Omega), \int_\Omega q_h = 0, q_h|_T \text{is constant},\ \forall\, T \in \mathcal{T}_h\}$. Prove the inf-sup condition:

$$\inf_{q_h\in\tilde{Q}_h}\sup_{\mathbf{v}_h\in\mathbf{V}_{h0}^{\rm CR}}\frac{(q_h,{\rm div}_h\mathbf{v}_h)}{|\mathbf{v}_h|_{1,h}\|q_h\|_{0,\Omega}} \gtrsim 1. \tag{23.35}$$

12. Consider the following Stokes problem

$$\begin{cases} -\Delta\mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega,\\ {\rm div}\mathbf{u} = 0 \text{ in } \Omega,\\ \mathbf{u} = \mathbf{0} \text{ on } \partial\Omega. \end{cases} \tag{23.36}$$

where the domain $\Omega = [0,1]\times[0,1]$. Please use the $P_2^0 - P_0^{-1}$ element to discretize this problem with the uniform mesh ($h = 0.25, 0.125, 0.0625, 0.03125$), and solve the relevant linear system. Treat the numerical solution u_h and p_h on the mesh with $h = 0.03125$ as the "exact" solution to compute the L^2 for p_h and H^1 error for u_h on other meshes, and write down a report.
13. (reserve) Prove the Cea lemma and Strang lemma.
14. (reserve) Prove the inf-sup condition given the existence of Fortin operator.

23.8 Adaptive finite element methods

1. Let $\mathcal{T}_{k+1}$ be a conforming and shape regular triangulation which is refined from a conforming and shape regular triangulation $\mathcal{T}_k$. Let u_{k+1} and u_k be solutions of the model Poisson equation in linear finite element space $\mathcal{V}_{k+1}$ and $\mathcal{V}_k$ respectively. Show the following orthogonality
$$\|u - u_{k+1}\|_a^2 = \|u - u_k\|_a^2 - \|u_{k+1} - u_k\|_a^2$$
2. Let $\mathcal{T}_{k+1}$ be a conforming and shape regular triangulation which is refined from a conforming and shape regular triangulation $\mathcal{T}_k$ using Döruler marking strategy by using a given $\theta \in (0, 1)$. Let u_{k+1} and u_k be solutions of the model Poisson equation in linear finite element space $\mathcal{V}_{k+1}$ and $\mathcal{V}_k$ respectively. Show the following contraction of the error estimator:
$$\eta^2(u_k, \mathcal{T}_k + 1) \le \rho\eta^2(u_k, \mathcal{T}_k),$$
where $\rho \in (0, 1)$ depends only on the shape regularity of $\mathcal{T}_k$ and the parameter θ.
3. Let τ be a non-degenerated triangular (i.e. $\alpha_0 \le \frac{|e|}{|e'|} \le \beta_0$) with vertices x_1, x_2, and x_3.
   a) Show that
   $$\int_\tau v^2 \cong h^2\left(v^2(x_1) + v^2(x_2) + v^2(x_3)\right)$$
   b) Show that
   $$\int_\tau |\nabla v|^2 \lesssim h^{-2}\int_\tau v^2,$$
   and
   $$\int_\Omega |\nabla v|^2 \lesssim h^{-2}\int_\Omega v^2.$$
4. Let A be the stiffness matrix of the model Poisson problem discretized on a conforming and shape regular triangulation in two dimensions by using linear finite element method. Show that the condition number of A is bounded as follows
$$\kappa(A) \le CN(1 + |\log(Nh_{\min}(N)^2)|),$$
where N is the number of degrees of freedom and $h_{\min} = \min\{h_\tau : \tau \in \mathcal{T}\}$.

23.9 Jacobi and Gauss-Seidel methods for consistently ordered matrices

1. The Gauss-Seidel iterative scheme for solving linear system $\mathbf{Ax} = \mathbf{b}$ with $\mathbf{A} \in \mathbb{R}^{N\times N}$ and $\mathbf{x}, \mathbf{b} \in \mathbb{R}^N$ is defined by
$$\mathbf{x}_i^{(m)} = \frac{1}{\mathbf{A}_{ii}}(\mathbf{b}_i - \sum_{j<i}\mathbf{A}_{ij}\mathbf{x}_j^{(m)} - \sum_{j>i}\mathbf{A}_{ij}\mathbf{x}_j^{(m-1)}), \quad i = 1 : N,$$
where $\left\{\mathbf{x}^{(m)}\right\}$, $m = 0, 1, \ldots,$ is the iterative sequence..
   a) Decompose $\mathbf{A} = \mathbf{D} + \mathbf{L} + \mathbf{U}$, where $\mathbf{D}$ is a diagonal matrix, $\mathbf{L}$ is a lower triangle matrix, and $\mathbf{U}$ is an upper triangle matrix. Show that the Gauss-Seidel iterative scheme can be written in the form
   $$\mathbf{x}^{(m)} = \mathbf{x}^{(m-1)} + (\mathbf{D} + \mathbf{L})^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}^{(m-1)}).$$
   b) For $\omega \in \mathbb{R}_+$, define the modified Gauss-Seidel iterative scheme:
   $$\mathbf{x}^{(m)} = \mathbf{x}^{(m-1)} + (\omega^{-1}\mathbf{D} + \mathbf{L})^{-1}(\mathbf{b} - \mathbf{A}\mathbf{x}^{(m-1)}).$$
   Assume $\mathbf{A}$ is symmetric positive definite. Prove that the modified Gauss-Seidel iterative scheme converges for any initial guess $\mathbf{x}^{(0)}$, if and only if $0 < \omega < 2$.

2. Let V be a linear vector space, with $(\cdot,\cdot)$ an inner product defined thereon. Let $A: V \to V$ be an SPD operator, and define an alternative inner product $(\cdot,\cdot)_A := (A\cdot,\cdot)$ on V. For an operator $C: V \to V$, define its adjoint operator with respect to $(\cdot,\cdot)$ by $C^t: V \to V$, such that

$$(C^t w, v) = (w, Cv), \ \forall\, w, v \in V,$$

and its adjoint operator with respect to $(\cdot,\cdot)_A$ by $C^*: V \to V$, such that

$$(C^* w, v)_A = (w, Cv)_A, \ \forall\, w, v \in V.$$

   a) For an operator $B: V \to V$, show that

$$\rho(I - \overline{B}A) = \rho((I - BA)^*(I - BA)),$$

   where $\overline{B} = B^t + B - B^t AB$, and $\rho(\cdot)$ means the spectral radius of an operator.

   b) For an operator $C: V \to V$, define its $A-$norm by

$$\|C\|_A := \sup_{v \in V\setminus\{0\}} \frac{\|Cv\|_A}{\|v\|_A},$$

   with $\|v\|_A^2 = (Av, v)$ for $v \in V$. Show that

$$\rho((I - BA)^*(I - BA)) = \|(I - BA)^*(I - BA)\|_A = \|I - BA\|_A^2.$$

3. Consider solving the following Poisson problem on a unit square $\Omega = [0,1] \times [0,1]$

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

by finite difference method (standard 5-point stencil) on uniform mesh with meshsize h. Denote the resulting linear system by $Ax = b$.

   a) Use Gauss Seidel method to solve the resulting linear system. Stopping criterion is $\frac{\|b-Ax^k\|}{\|b-Ax^0\|} \le 10^{-8}$ and the initial guess is the zero vector. Make a table to record the number of iterations for different h.

   b) Use SOR method to solve the resulting linear system. Stopping criterion is $\frac{\|b-Ax^k\|}{\|b-Ax^0\|} \le 10^{-8}$ and the initial guess is the zero vector. Make a table to record the number of iterations for different h and the relaxation parameter ω. Numerically determine the best ω

   c) Use the SOR method to solve the resulting linear system with theoretically best relaxation parameter

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(D^{-1}A)^2}},$$

   where D is the diagonal of A and $\rho(M)$ denote the spectral radius of matrix M. Stopping criterion is $\frac{\|b-Ax^k\|}{\|b-Ax^0\|} \le 10^{-8}$ and the initial guess is the zero vector. Make a table to record the number of iterations.

   d) Use conjugate gradient method to solve the resulting linear system. Stopping criterion is $\frac{\|b-Ax^k\|}{\|b-Ax^0\|} \le 10^{-8}$ and the initial guess is the zero vector. Make a table to record the number of iterations for different h.

4. Consider using subspace correction method to solve $Au = f$,

   a) Show that the parallel subspace correction method for $Au = f$ is equivalent to the modified Jacobi method for $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:

$$\underset{\sim}{u}^m = \underset{\sim}{u}^{m-1} + \underset{\sim}{R}(\underset{\sim}{f} - \underset{\sim}{A}\underset{\sim}{u}^{m-1})$$

b) Show that the successive subspace correction method for $Au = f$ is equivalent to the modified G-S method for $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:

$$\underset{\sim}{u}^m = \underset{\sim}{u}^{m-1} + (\underset{\sim}{R}^{-1} + \underset{\sim}{L})^{-1}(\underset{\sim}{f} - \underset{\sim}{A}\underset{\sim}{u}^{m-1}).$$

5. Assume that $\widetilde{A}$ is a block tridiagonal matrix with invertible diagonal block and $\omega \neq 0$. Then if λ is a nonzero eignvalue of $M_{SOR} = (D - \omega E)^{-1}(\omega F + (1-\omega)D)$, any scalar μ that satisfies

$$\lambda + \omega - 1 = \lambda^{1/2}\omega\mu$$

is an eignvalue of M_J, which is defined as $M_J = I - D^{-1}\widetilde{A}$. If $\widetilde{A}$ is symmetric so that M_J only has real eignvalues, prove that

a) $\rho(M_{SOR}) < 1$ iff $\rho(M_J) < 1$ and $0 < \omega < 2$.

b) If SOR converges,

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho(M_J)^2}} = \arg\min_{\omega} \rho(M_{SOR}(\omega)),$$

and

$$\min_{\omega} \rho(M_{SOR}(\omega)) = \frac{1 - \sqrt{1 - \rho(M_J)^2}}{1 + \sqrt{1 - \rho(M_J)^2}}.$$

6. Consider solving the linear system of equations

$$\mathsf{A}\mathsf{u} = \mathsf{f},$$

by the method of subspace correction. Here $\mathsf{A} \in \mathbb{R}^{n\times n} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is symmetric positive definite.

a) Consider the simple decomposition of $\mathbb{R}^n$

$$\mathbb{R}^n = \sum_{i=1}^{n} \operatorname{span}\{e_i\}, \tag{23.37}$$

where e_i is the i-th column of the identity matrix. Show that the parallel subspace correction method is just the Jacobi iterative method.

b) Consider the same decomposition (23.64), show that the successive subspace correction method is just the Gauss-Seidel iterative method.

7. Consider the Poisson equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

discretized by linear finite element method on a uniform triangulation grid (see Figure 6) of $\Omega = [0,1]\times[0,1]$ with mesh size h. Here, we have

$$0 = x_0 < x_1 < \cdots < x_{n+1} = 1, \quad x_j = \frac{j}{n+1}, \quad (j = 0, \cdots, n+1).$$

and

$$0 = y_0 < y_1 < \cdots < y_{n+1} = 1, \quad y_j = \frac{j}{n+1}, \quad (j = 0, \cdots, n+1).$$

The weak form is: Find $u_h \in V_h$, such that

$$a(u_h, v_h) = (f, v_h), \ \forall\, v_h \in V_h \tag{23.38}$$

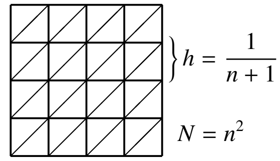

Fig. 23.8. Two-dimensional uniform grid for finite element method

where $a(u_h, v_h) = \int_\Omega \nabla u_h \nabla v_h$ and $(f, v_h) = \int_\Omega f v_h$, and V_h is the corresponding linear finite element space. Take $v_h = \phi^h_{i,j}$ where $\phi^h_{i,j}$ is the basis function at point (x_i, y_j), from (23.67), we have

$$4\mathsf{u}_{i,j} - \mathsf{u}_{i-1,j} - \mathsf{u}_{i+1,j} - \mathsf{u}_{i,j-1} - \mathsf{u}_{i,j+1} = \mathsf{f}_{i,j},$$

where $\mathsf{f}_{i,j} = (f, \phi^h_{i,j})$. This leads to the following linear system of equations.

$$\mathsf{Au} = \mathsf{f}.$$

where $\mathsf{A} = \text{tridiag}(-\mathsf{I}, \mathsf{B}, -\mathsf{I})$ and $\mathsf{B} = \text{tridiag}(-1, 4, -1)$ and $\mathsf{u} = (\mathsf{u}_{i,j})$ and $\mathsf{f} = (\mathsf{f}_{i,j})$ with i and j both follow the lexicographic ordering.
Implement the following Gauss-Seidel method:

Gauss-Seidel method: $[\mathsf{u}] = \mathtt{GS}(\mathsf{u}, \mathsf{f}, n)$

Given an initial guess u^0, consider the Gauss-Seidel iteration as follows:

Choose right hand side f and initial guess u^0 freely (constant right hand side and random initial guess are recommended), and set the tolerance to be $\mathtt{Tol} = 10^{-6}$. Make a table to report the number of iterations, convergence factor ($\|\mathsf{f} - \mathsf{A}\mathsf{u}^k\| / \|\mathsf{f} - \mathsf{A}\mathsf{u}^{k-1}\|$), and CPU time for $h = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}$.

8. Following problem 2, we use two-grid method to solve the resulting linear system of equations. For the find grid, we choose $h = 2^{-J}$ with a given integer J and $n = 2^J - 1$. For the coarse grid, we choose $H = 2h = 2^{-(J-1)}$ and $n_H = 2^{J-1} - 1$. In the implementation of the two-grid method, we do not need to form A explicitly, what we need is just the action of A on a vector u, i.e. $\mathsf{v} = \mathsf{Au}$. The following algorithm shows the action of A

Action of A: $[\mathsf{v}] = \mathtt{Action}(\mathsf{u}, n)$

The restriction and prolongation are implemented as follows:

Restriction: $[\mathsf{r}^H] = \mathtt{restrict}(\mathsf{r}^h, n_H)$

Prolongation: $[\mathsf{e}^h] = \mathtt{prolongate}(\mathsf{e}^H, n_H)$

Now, one step of the two-grid method with GS smoother is defined as follows:

Two-grid method: $[\mathsf{u}] = \mathtt{TwoGrid}(\mathsf{u}, \mathsf{f}, n, n_H)$

Given an initial guess u^0, please implement the two-grid iteration as follows:

Choose right hand side f and initial guess u^0 freely (constant right hand side and random initial guess are recommended), and set the tolerance to be $\mathtt{Tol} = 10^{-6}$. Make a table to report the number of iterations, convergence factor ($\|\mathsf{f} - \mathsf{A}\mathsf{u}^k\| / \|\mathsf{f} - \mathsf{A}\mathsf{u}^{k-1}\|$), and CPU time for $J = 3, 4, 5, 6$.

9. *(Optional)* In this problem, we use multigrid method to linear system of equations. MG method is based on the following nested spaces
$$V_1 \subset V_2 \subset \cdots \subset V_J,$$
where V_l, $l = 1, 2, \cdots, J$, are linear finite element spaces on uniform grids with mesh size $h_l = 2^{-l}$ and $n_l = 2^l - 1$. Recursive implementation of the MG method with GS smoother is defined as follows:

   MG method: $[\mathsf{u}_l] = \mathtt{MultiGrid}(\mathsf{u}_l, \mathsf{f}_l, l)$

   Given an initial guess u^0, please implement the MG iteration as follows:

   Choose right hand side f and initial guess u^0 freely (constant right hand side and random initial guess are recommended), and set the tolerance to be $\mathtt{Tol} = 10^{-6}$. Make a table to report the number of iterations, convergence factor ($\|\mathsf{f} - \mathsf{A}\mathsf{u}^k\| / \|\mathsf{f} - \mathsf{A}\mathsf{u}^{k-1}\|$), and CPU time for $J = 3, 4, 5, 6$.
10. Prove that Jacobi method converges for any 2×2 symmetric positive definite matrices.

23.10 Finite difference method

Exercise 1. Prove that
$$u'(x) = \frac{u(x + \frac{h}{2}) - u(x - \frac{h}{2})}{h} + O(h^2),$$
$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + O(h^2).$$

Exercise 2. Consider the domain $(0, 1)$, $x_i = ih$, $0 \le i \le N$ and $h = 1/N$. Let $\{\phi_j\}_{j=1}^N$ be those piecewise linear polynomials which is continuous at internal points and satisfy that
$$\phi_j(x_i) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}.$$
Prove that
$$u_h(x) = \sum_{j=1}^N u_h(x_j)\phi_j(x), \ \forall u_h \in V_h.$$

Exercise 3. Consider the domain $(0, 1)$, $x_i = ih$, $0 \le i \le N$ and $h = 1/N$. Basis functions $\{\phi_j\}_{j=1}^N$ are defined in Exercise 2. Prove that
$$\int_0^1 \phi'_{j-1}\phi'_j = \int_0^1 \phi'_j\phi'_{j+1} = -\frac{1}{h}, \quad 1 < j < N$$
$$\int_0^1 (\phi'_j)^2 = \frac{2}{h}, \quad 1 \le j \le N.$$
Therefore, the finite element equation is reduced to
$$\frac{-\mu_{j-1} + 2\mu_j - \mu_{j+1}}{h} = f_j, \quad 1 \le j \le N, \tag{23.39}$$
where $f_j = \int_0^1 f\phi_j$. Thus, the finite element method and the finite difference method give the same equation.

Exercise 4. For the 1D Poisson equation in (0, 1)

$$\begin{cases} -u'' = f \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{23.40}$$

define the operators

$$(\mathcal{L}u)(x) = -u''(x), \quad (\mathcal{L}_h u)(x) = \frac{-u(x+h) + 2u(x) - u(x-h)}{h^2},$$

prove that for smooth enough u,

$$\mathcal{L}u - \mathcal{L}_h u = O(h^2).$$

Furthermore, let u_h be the solution of $(\mathcal{L}_h u_h)(x_i) = f(x_i)$ with $u_h(0) = u_h(1) = 0$. Suppose the solution u and source term f are smooth enough, then

$$\max_i |u(x_i) - u_h(x_i)| = O(h^2).$$

Exercise 5. For the 1D Poisson equation in (0, 1)

$$\begin{cases} -u'' = f \quad \text{in } (0,1), \\ u(0) = u(1) = 0, \end{cases} \tag{23.41}$$

let u_h be the solution of $(\mathcal{L}_h u_h)(x_i) = f(x_i)$ with $u_h(0) = u_h(1) = 0$. Suppose the solution u and source term f are smooth enough, then

$$\max_i |u(x_i) - u_h(x_i)| = O(h^2). \tag{23.42}$$

Now we set $u = \sin \pi x$, and then $f = -\pi^2 \sin \pi x$. Implement the finite difference method and verify the above convergence rate shown in (23.42).

23.11 Basic finite element methods

Exercise 6. Let w be a locally integrable function in Ω (denoted by $L_{\text{loc}}(\Omega)$), prove that

$$w = 0 \ \text{ a.e.} \iff \int_\Omega w\varphi = 0, \forall \varphi \in C_0^\infty(\Omega).$$

Exercise 7. Derive the finite element formulation for the 1D Poisson equation (4.11) on non-uniform grid.

Exercise 8. Consider two different triangulations on the unit square $\Omega = (0,1)^2$: the uniform grid (Fig. 23.9) and the criss-cross grid (Fig. 23.10).

(a) For the uniform grid, let $u_h = \sum_{ij} u_{i,j}\phi_{i,j}$, where $\phi_{i,j}$ is the nodal basis function associated with the grid point (x_i, y_i). Then the finite element formulation

$$a(u_h, \phi_{ij}) = (f, \phi_{ij})$$

can be written as

$$\frac{-u_{i,j-1} - u_{i,j+1} + 4u_{i,j} - u_{i-1,j} - u_{i+1,j}}{h^2} = \frac{1}{h^2}\tilde{f}_{i,j}. \tag{23.43}$$

Here

$$\tilde{f}_{i,j} = \frac{1}{h^2}\int_\Omega f(x,y)\phi_{ij}(x,y).$$

(b) Prove that $\tilde{f}_{i,j} - f(x_i, y_i) = O(h^2)$ for the uniform grid.

(c) Derive the following finite element formulation on criss-cross grid in the form of (23.43):

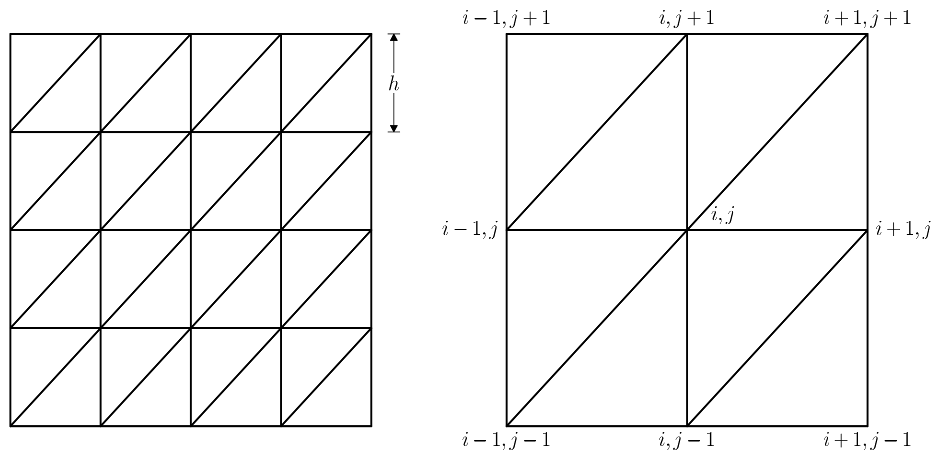

Fig. 23.9. *Uniform triangulation of the domain* $(0,1)^2$

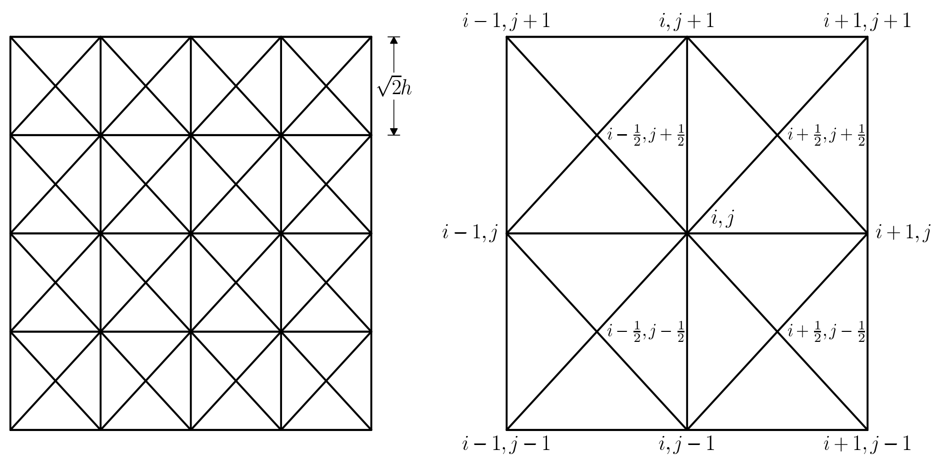

Fig. 23.10. *Criss-cross triangulation of the domain* $(0,1)^2$

- At the (x_i, y_j) node,

(23.44)
$$\begin{aligned}\frac{(\nabla u_h, \nabla \phi_{i,j})}{h^2} + \Delta u(x_i, y_i) &= \frac{-u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i-\frac{1}{2},j-\frac{1}{2}} + 4u_{i,j} - u_{i+\frac{1}{2},j-\frac{1}{2}} - u_{i-\frac{1}{2},j+\frac{1}{2}}}{h^2} + \Delta u(x_i, y_i) = \mathcal{O}(h^2),\\ \frac{(f, \phi_{i,j})}{h^2} - f(x_i, y_i) &= \tilde{f}_{i,j} - f(x_i, y_i) = \frac{1}{3} f(x_i, y_j) + \mathcal{O}(h^2).\end{aligned}$$

- At the node $(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}})$,

(23.45)

$$\frac{(\nabla u_h, \nabla\phi_{i+\frac{1}{2},j+\frac{1}{2}})}{h^2} + \Delta u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = \frac{-u_{i,j} - u_{i+1,j+1} + 4u_{i+\frac{1}{2},j+\frac{1}{2}} - u_{i,j+1} - u_{i+1,j}}{h^2} + \Delta u(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = O(h^2),$$

$$\frac{(f, \phi_{i+\frac{1}{2},j+\frac{1}{2}})}{h^2} - f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = \tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} - f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) = -\frac{1}{3}f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}) + O(h^2).$$

(d) Prove that for the criss-cross grid

$$\tilde{f}_{i,j} \not\to f(x_i, y_j), \quad \tilde{f}_{i+\frac{1}{2},j+\frac{1}{2}} \not\to f(x_{i+\frac{1}{2}}, y_{j+\frac{1}{2}}).$$

Namely, the corresponding finite element scheme is not consistent with the finite difference scheme in classic case.

(e) Find the sufficient condition for the triangulation such that the corresponding finite element scheme is consistent with the finite difference scheme.

Exercise 9. Let τ be a d-dimensional simplex with vertices a_i and $\lambda_i(x)$ be the corresponding barycentric coordinate. Prove that

$$\lambda_i(x) = \frac{|\tau_i|}{|\tau|},$$

where $|\tau_i|$ is the volume of the sub-simplex $\triangle_{a_1\cdots a_{i-1}xa_{i+1}\cdots a_{d+1}}$.

Exercise 10. Suppose Ω is bounded, let $v \in H^1_{\Gamma_D} = \{u \in H^1(\Omega), u|_{\Gamma_D} = 0, |\Gamma_D| > 0\}$, then

$$\|v\|_{0,\Omega} \le C(\Omega)\|\nabla v\|_{0,\Omega}.$$

Exercise 11. (Dual of $H^1_0(\Omega)$) We first give the L^2 pairing for $f \in L^2(\Omega)$ as

$$\langle f, v\rangle = \int_\Omega fv,$$

then define the norm $\|\cdot\|_{-1,\Omega}$ as

$$\|f\|_{-1,\Omega} = \sup_{v\in H^1_0} \frac{\int_\Omega fv}{|v|_{1,\Omega}}.$$

Prove that

$$H^{-1}(\Omega) = \overline{L^2(\Omega)}^{\|\cdot\|_{-1,\Omega}}$$

is a representation of $(H^1_0)^*$.

Exercise 12. Assume that $\Omega \subseteq \mathbb{R}^d$, $\hat{\Omega} \subseteq \mathbb{R}^d$ are two open bounded domains. And $F : \hat{\Omega} \to \Omega$ is an affine map s.t. $x = F(\hat{x}) = B\hat{x} + x_0$, where $x_0, x \in \Omega$, $\hat{x} \in \hat{\Omega}$. For a given function $\hat{u}(\hat{x})$, we define $u(x) = \hat{u}(F^{-1}(x))$. Note that for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_d)$ with $|\alpha| = \sum_{i=1}^d \alpha_i = m$, we define a multi-linear m-form as

$$[D^m_x u(x)](\underbrace{e_1, e_1, \cdots, e_1}_{\alpha_1}, \underbrace{e_2, e_2, \cdots, e_2}_{\alpha_2}, \cdots, \underbrace{e_d, e_d, \cdots, e_d}_{\alpha_d}) = \frac{\partial^\alpha}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}},$$

where e_i $(i = 1, \cdots, d)$ is the standard set of orthonormal basis of $\mathbb{R}^d$. It can be verified that when u is smooth enough, for a given index set $\mathcal{I} = \{i_1, i_2, \cdots, i_m\}$ that satisfies

1. $i_k \in \{1, 2, \cdots, d\}$ $(k = 1, 2, \cdots, m)$,

2. there are α_1 indices in $\mathcal{I}$ equal to 1, α_2 indices in $\mathcal{I}$ equal to 2, and so on. That is, there are α_k indices in $\mathcal{I}$ equal to k ($k = 1, 2, \cdots, d$),

we have

$$[D_x^m u(x)](e_{i_1}, e_{i_2}, \cdots, e_{i_m}) = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

This is true because when function u is smooth enough, you can exchange the order of partial derivatives.

1. Prove that $[D_x u(x)](\xi_1) = \nabla_x u(x) \cdot \xi$, for any $\xi \in \mathbb{R}^d$.
2. From the above description, we can conclude that $\hat{u}(\hat{x}) = u(F(x))$, prove that

$$[D_x u(x)](B\xi) = [D_{\hat{x}} \hat{u}(\hat{x})](\xi), \ \forall \xi \in \mathbb{R}^d.$$

3. Prove that

$$[D_x^m u(x)](B\xi_1, B\xi_2, \cdots, B\xi_m) = [D_{\hat{x}}^m \hat{u}(\hat{x})](\xi_1, \xi_2, \cdots, \xi_m),$$

   for any vectors $\xi_i \in \mathbb{R}^d$, $1 \le i \le m$.
4. Prove that

$$|u|_{m,K} \le c(\det B)^{1/2} \|B^{-1}\|^m |\hat{u}|_{m,\hat{K}},$$
$$|\hat{u}|_{m,\hat{K}} \le c(\det B)^{-1/2} \|B\|^m |u|_{m,K},$$

   where K is an element in Ω, and $\hat{K}$ is an element in $\hat{\Omega}$. And c is a constant that only depends on m and the space dimension d.

Exercise 13. Assume that

$$K = \text{triangle} = [a_0, a_1, a_2] \subseteq \mathbb{R}^2,$$
$$\mathcal{P} = P_2(K) = \{p(x) : \ p(x) \text{ is a polynomial on K and its order} \le 2\},$$
$$\mathcal{N}(v) = \{v(a_1) : \ 0 \le i \le 2\} \cup \left\{v(a_{ij}) : \ 0 \le i < j \le 2\right\}, \ a_{ij} = \frac{1}{2}(a_i + a_j).$$

Prove that the triple $(K, \mathcal{P}, \mathcal{N})$ is a well-defined finite element space.

23.12 Sobolév spaces and finite element spaces

Exercise 14. Corresponding to the triangulations $\mathcal{T}_h$, the finite element spaces $\mathcal{S}^h = \mathcal{S}^h(\Omega)$ is defined by

$$\mathcal{S}^h(\Omega) = \{v \in C(\bar{\Omega}) : v|_\tau \in \mathcal{P}_1(\tau), \quad \forall \ \tau \in \mathcal{T}_h\}.$$

It is easy to see that $\mathcal{S}^h(\Omega) \subset H^1(\Omega)$. We further define that

$$\mathcal{S}_0^h(\Omega) = \mathcal{S}^h(\Omega) \cap H_0^1(\Omega).$$

1. Show that $\mathcal{S}^h$ is indeed a subspace of $H^1(\Omega)$.
2. Assume that $\partial\Omega$ is curved but piecewise convex. Define

$$\mathcal{S}^h = \{v : v \in S^h(\bar{\Omega}_h), v|_{\Omega \setminus \Omega_h} \text{ is the natural linear extention}\} \tag{23.46}$$

   and

$$\mathcal{S}_0^h = \{v : v \in S_0^h(\bar{\Omega}_h), v|_{\Omega \setminus \Omega_h} = 0\}. \tag{23.47}$$

   Prove that $\mathcal{S}^h \subset H^1(\Omega)$ and $\mathcal{S}_0^h \subset H_0^1(\Omega)$.

3. Prove, for the space $\mathcal{S}^h$ defined as above,

$$\|v\|_{0,\Omega\setminus\Omega_h} \lesssim h\|v\|_1 \quad \forall\ v \in \mathcal{S}^h.$$

Exercise 15. Assume that $\Omega \subseteq \mathbb{R}^d$ is an open bounded domain, and that $\Omega = \Omega_1 \cup \Omega_2$. Ω_1 and Ω_2 are both open, and $\Omega_1 \cap \Omega = \emptyset$, $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \Gamma$. Given a function $v \in [C^1(\bar{\Omega}_i)]^3$, $i = 1, 2$, prove that

1. $v \in H(\mathrm{curl}, \Omega)$ if $v \times n$ is continuous across Γ.
2. $v \in H(\mathrm{div}, \Omega)$ if $v \cdot n$ is continuous across Γ.

Exercise 16. Consider the finite element space for $H(\mathrm{curl})$

$$\begin{aligned}
&K = \text{tetrahedron} = [a_0, a_1, a_2, a_3] \subseteq \mathbb{R}^3,\\
&\mathcal{P}(K) = \left\{\boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{x},\ \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^3, \mathbf{x} \in K\right\},\\
&\mathcal{N}(v) = \left\{\int_{e_{ij}} v \cdot \boldsymbol{\tau}_{e_{ij}},\ e_{ij} = [a_i, a_j], 0 \le i < j \le 3,\ \boldsymbol{\tau}_{e_{ij}} \text{ is the unit tangential vector of edge } e_{ij}\right\}.
\end{aligned}$$

Please solve the following problems.

1. Prove that $\mathcal{P}(K)$ is unisolvent with respect to $\mathcal{N}$.
2. Construct the nodal basis in terms of barycentric coordinates for this given $\mathcal{N}$.

Exercise 17. The finite element triple for $H(\mathrm{curl})$ is defined as

$$\begin{aligned}
&K = \text{tetrahedron} = [a_0, a_1, a_2, a_3] \subseteq \mathbb{R}^3,\\
&\mathcal{P}(K) = \left\{\boldsymbol{\alpha} + \boldsymbol{\beta} \times \mathbf{x},\ \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^3, \mathbf{x} \in K\right\},\\
&\mathcal{N}(v) = \left\{\int_{e_{ij}} v \cdot \boldsymbol{\tau}_{e_{ij}},\ e_{ij} = [a_i, a_j], 0 \le i < j \le 3,\ \boldsymbol{\tau}_{e_{ij}} \text{ is the unit tangential vector of edge } e_{ij}\right\}.
\end{aligned}$$

And the finite element triple for $H(\mathrm{div})$ is defined as

$$\begin{aligned}
&K = \text{tetrahedron} = [a_0, a_1, a_2, a_3] \subseteq \mathbb{R}^3,\\
&\mathcal{P}(K) = \left\{\boldsymbol{\alpha} + \beta\mathbf{x} :\ \boldsymbol{\alpha} \in \mathbb{R}^3,\ \beta \in \mathbb{R},\ \mathbf{x} \in K\right\},\\
&\mathcal{N}(v) = \left\{\int_{F_{ijk}} v \cdot \mathbf{n}_{F_{ijk}},\ F = [a_i, a_j, a_k],\ 0 \le i < j < k \le 3,\ \mathbf{n}_{F_{ijk}} \text{ is the unit outer normal vector of face } F_{ijk}\right\}.
\end{aligned}$$

Glue these finite element triples together to get the conforming finite element spaces, i.e. $H_h(\mathrm{curl}) \subseteq H(\mathrm{curl})$, and $H_h(\mathrm{div}) \subseteq H(\mathrm{div})$.

23.13 Mixed Methods

Exercise 18. Let U and V be two Hilbert spaces, with inner products $(\cdot,\cdot)_U$ and $(\cdot,\cdot)_V$ respectively. Let $\mathcal{B}(\cdot,\cdot) : U \times V \to \mathbb{R}$ be a continuous bilinear form

$$\mathcal{B}(u, v) \le \|\mathcal{B}\|\|u\|_U\|v\|_V. \tag{23.48}$$

Assume the inf-sup condition holds:

$$\inf_{u\in U} \sup_{v\in V} \frac{\mathcal{B}(u, v)}{\|u\|_U\|v\|_V} = \inf_{v\in V} \sup_{u\in U} \frac{\mathcal{B}(u, v)}{\|u\|_U\|v\|_V} = \alpha > 0. \tag{23.49}$$

Let $U_h \subset U$ and $V_h \subset V$ be two nontrivial subspaces of U and V respectively, and the discrete inf-sup condition holds:

$$\inf_{u_h\in U_h}\sup_{v_h\in V_h}\frac{\mathcal{B}(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} = \inf_{v_h\in V_h}\sup_{u_h\in U_h}\frac{\mathcal{B}(u_h,v_h)}{\|u_h\|_U\|v_h\|_V} = \alpha_h > 0. \tag{23.50}$$

Consider the following variational problem: Find $u \in U$ such that

$$\mathcal{B}(u,v) = \langle f, v\rangle,\ \forall\, v \in V, \tag{23.51}$$

where $f \in V^*$ and $\langle\cdot,\cdot\rangle$ is the pairing between V^* and V, and its discretisation: find $u_h \in U_h$, such that

$$\mathcal{B}(u_h,v_h) = \langle f, v_h\rangle,\ \forall\, v_h \in V_h. \tag{23.52}$$

Prove

$$\|u-u_h\|_U \le \frac{\|\mathcal{B}\|}{\alpha_h}\inf_{w_h\in U_h}\|u-w_h\|_U. \tag{23.53}$$

Exercise 19. Let $\Omega \in \mathbb{R}^d$ be a connected domain with a Lipschitz boundary. Suppose we have the following Korn's inequality,

$$\|u\|_1 \lesssim \|\epsilon(u)\|_1 + \|u\|_0, \quad \forall u \in (H_0^1)^d.$$

Use the standard contradiction argument and compact embedding to prove:

$$\|u\|_1 \lesssim \|\epsilon(u)\|_0, \quad \forall u \in (H_D^1)^d. \tag{23.54}$$

where $\mathrm{meas}(\Gamma_D) \neq 0$.

Exercise 20. Consider the nodal value interpolation operator

$$I_h v = \sum_{i=1}^{n} v(x_i)\phi_i,$$

when $x \in \mathbb{R}^2$, is it true that

$$\|v - I_h v\|_{0,\Omega} \le Ch|v|_{1,\Omega}, \quad \forall v \in H^1(\Omega)?$$

Please prove this inequality if it is true, and disprove it otherwise.

Exercise 21. Assume that $\left\{\phi_i^*\right\} \subset V_h$, V_h is a space of piecewise linear functions. And

$$\langle \phi_j^*, \phi_i\rangle = (\phi_j^*, \phi_i)_{L^2(\Omega)} = \delta_{ij}.$$

Then for any $v_h \in V_h$, $v_h = \sum\limits_{i=1}^{n} v_h(x_i)\phi_i$, we have

$$\langle \phi_j^*, v_h\rangle = v_h(x_j).$$

Define interpolation $\Pi_h v = \sum\limits_{i=1}^{n} (\phi_i^*, v)_{L^2(\Omega)}\phi_i$. Prove that

1. $\|\Pi_h v\|_{L^2} \le C\|v\|_{L^2}$, for any $v \in L^2(\Omega)$.
2. $\|v - \Pi_h v\|_{L^2} \le Ch|v|_{1,\Omega}$, for any $v \in H^1(\Omega)$.

Exercise 22. Let V and Q be two Hilbert spaces, with inner products $(\cdot,\cdot)_V$ and $(\cdot,\cdot)_Q$ respectively. Let bilinear forms $a(\cdot,\cdot): V\times V\mapsto\mathbb{R}$ and $b(\cdot,\cdot): V\times Q\mapsto\mathbb{R}$ satisfy

$$\begin{aligned} a(u,v) &\le C_a\|u\|_V\|v\|_V \\ b(u,p) &\le C_b\|u\|_V\|p\|_Q \\ a(u,u) &\ge \gamma_a\|u\|_V^2 \\ \sup_{v\in V}\frac{b(v,p)}{\|v\|_V} &\ge \gamma_b\|p\|_Q, \end{aligned}$$

for any $u,v\in V$ and $p\in Q$. Moreover, assume that $a(\cdot,\cdot)$ is symmetric. Prove $L(u,p;v,q) := a(u,v)+b(u,p)+b(v,q)$ satisfies that for any $u\in U$ and $p\in Q$

$$\sup_{v\in V, q\in Q}\frac{L(u,p;v,q)}{(\|v\|_V^2+\|q\|_Q^2)^{1/2}} \ge C_L(\|u\|_V^2+\|p\|_Q^2)^{1/2}$$

and write C_L in terms of C_a, C_b, γ_a and γ_b.

Exercise 23. Consider variational form of Stokes problem: find $(u,p)\in V\times Q=[H_D^1(\Omega)]^d\times L^2(\Omega)$ such that

$$\begin{cases} a(u,v)+b(v,p)=\langle f,v\rangle, \\ b(u,q)=0, \end{cases} \tag{23.55}$$

where $a(u,v)=2(\epsilon(u),\epsilon(v))$, and $b(v,q)=-(\nabla\cdot v,q)$. Prove that the solution to problem (23.55) satisfies

$$u=\arg\min_{\nabla\cdot v=0}\left(\frac{1}{2}a(v,v)-\langle f,v\rangle\right).$$

Exercise 24. Consider the model problem:

$$\begin{cases} -\nabla\cdot(\kappa\nabla p)=g, \ \ in\ \Omega, \\ p=0, \ \ on\ \partial\Omega. \end{cases}$$

1. Prove that this probably is equivalent to the following formulation: find $(u,p)\in V\times Q=H(\mathrm{div})\times L^2$, such that

$$\begin{cases} (\kappa^{-1}u,v)-(\nabla\cdot v,p)=0, \\ -(\nabla\cdot u,q)=\langle g,q\rangle, \end{cases}$$

   for any $(v,q)\in V\times Q$. In particular, verify that the boundary condition $p=0$ is implicitly (or naturally) imposed in the above variational formulation.
2. It is proved (please refer to your class note for details) that this variational form is well-posed. That is, it satisfies the inf-sup condition. Prove that if $V_h=RT_0$, $Q_h=P_0^{-1}$, the finite element discretization: find $(u_h,p_h)\in V_h\times Q_h$, such that for any $(v_h,q_h)\in V_h\times Q_h$,

$$\begin{cases} (\kappa^{-1}u_h,v_h)-(\nabla\cdot v_h,p_h)=0, \\ -(\nabla\cdot u_h,q_h)=\langle g,q_h\rangle. \end{cases}$$

   satisfies the discrete inf-sup condition. RT_0 is a conforming finite element space for $H(\mathrm{div})$, whose finite element triple is given by

$K = \text{tetrahedron} = [a_0, a_1, a_2, a_3] \subseteq \mathbb{R}^3,$

$\mathcal{P}(K) = \left\{\boldsymbol{\alpha} + \beta\mathbf{x} : \ \boldsymbol{\alpha} \in \mathbb{R}^3, \ \beta \in \mathbb{R}, \ \mathbf{x} \in K\right\},$

$\mathcal{N}(v) = \left\{\int_{F_{ijk}} v \cdot \mathbf{n}_{F_{ijk}} : \ F = [a_i, a_j, a_k], \ 0 \le i < j < k \le 3, \ \mathbf{n}_{F_{ijk}} \text{ is the unit outer normal vector of face } F_{ijk}\right\}.$

And P_0^{-1} is the piecewise constant, which is also a conforming finite element space of $L^2(\Omega)$.

Exercise 25. Consider the model problem:

$$\begin{cases} -\nabla \cdot (\kappa \nabla p) = g, \ in \ \Omega, \\ \frac{\partial p}{\partial n} = 0, \ on \ \partial\Omega. \end{cases}$$

Identify the appropriate Sobolev spaces and derive a mixed variational formulation for the above problem.

Exercise 26. In 2D, assume that $u_h = (u_1, u_2)^T \in [H_0^1(\Omega)]^2$, where $\Omega = [0,1] \times [0,1]$. Uniform grid is used (figure 23.11). Both u_1 and u_2 are discretized by piecewise linear functions. Please verify that if $\nabla \cdot u_h = 0$, and $u_h = 0$ on $\partial\Omega$, $u_h = 0$ in Ω.

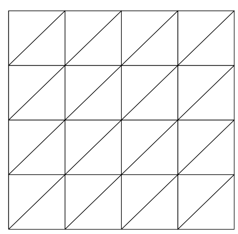

Fig. 23.11. Uniform grid in 2D

Hint: Please determine the value of u_h starting from the boundary.

Exercise 27. It can be proved that the problem: find $(u, p) \in V \times Q$ such that

$$\begin{cases} a(u, v) + b(v, p) = \langle f, v\rangle, \ \forall v \in V, \\ b(u, q) - c(p, q) = \langle g, q\rangle, \ \forall q \in Q, \end{cases}$$

is well-posed if the following conditions are satisfied:

$$a(u, v) \le C_1 \|u\|_V \|v\|_V, \tag{23.56}$$

$$c(pq) \le C_2 \|p\|_Q \|q\|_Q, \tag{23.57}$$

$$b(v, q) \le C_3 \|v\|_V \|q\|_Q, \tag{23.58}$$

$$a(v, v) \ge M_1 \|v\|_V, \ \forall v \in K = Ker(B), \tag{23.59}$$

$$c(q, q) \ge M_2 \|q\|_Q, \ \forall q \in H = Ker(B'), \tag{23.60}$$

$$\sup_{v \in V} \frac{b(v, q)}{\|v\|_V} \ge \beta \|q\|_Q, \ \forall q \in H^{\perp}. \tag{23.61}$$

Please answer the following questions

1. Whether the conditions (23.59) and (23.60) can be replaced by the corresponding inf-sup conditions?
2. Assume the answer to the above question is yes, are these conditions necessary for the well-posedness?

Exercise 28. Consider the mixed formulation of linear elasticity: find $(u, p) \in V \times Q$ such that

$$\begin{cases} \mu(\epsilon(u), \epsilon(v)) + (p, \mathrm{div} v) = \langle f, v\rangle, \ \forall v \in V, \\ (\mathrm{div} u, q) - \lambda^{-1}(p, q) = 0, \ \forall q \in Q. \end{cases}$$

Assume that $V = [H^1(\Omega)]^d$. If $\lambda \to \infty$, what will happen to the constants in the inf-sup conditions, namely, the constants in (23.56)-(23.61)?

23.14 eXtended Galerkin method

Exercise 29. Assume that $K \subset \mathbb{R}^d$ is an arbitrary polygon domain, with boundary ∂K. Define

$$S(K) = \left\{v = \{v_0, v_b\} : \ v_0 \in L^2(K), \ v_b \in L^2(\partial K)\right\}.$$

In weak Galerkin method, the weak gradient ∇_w of $v \in S(K)$ is defined by

$$\langle \nabla_w v, \boldsymbol{q}\rangle_K = -(v_0, \nabla \cdot \boldsymbol{q})_K + \langle v_b, \boldsymbol{q} \cdot \boldsymbol{n}\rangle_{\partial K}, \ \forall \boldsymbol{q} \in [H^1(K)]^d.$$

1. Define the weak curl and weak div operators similarly.
2. Define weak Laplacian operator $\Delta_w v$ by

$$\langle \Delta_w v, \phi\rangle_K = (v_0, \Delta\phi)_K - \langle v_b, \nabla\phi \cdot \boldsymbol{n}\rangle_{\partial K} + \langle \boldsymbol{v}_g \cdot \boldsymbol{n}, \phi\rangle_{\partial K}, \ \forall \phi \in H^2(K),$$

for any $v \in W(K)$. Here, $W(K)$ is defined by

$$W(K) = \left\{v = \{v_0, v_b, \boldsymbol{v}_g\} : \ v_0 \in L^2(K), \ v_b \in L^2(\partial K), \ \boldsymbol{v}_g \in [L^2(\partial K)]^d\}.\right\}.$$

Please verify that

$$\Delta_w v = \nabla_w \cdot \nabla_w v, \ \forall v \in W(K).$$

Exercise 30. Assume that $\mathcal{T}_h$ is a shape regular polygon partition of the domain Ω, and $\mathcal{E}_h^0$ is the set of all interior edges or faces of the mesh. Consider the Poisson equation with Dirichlet boundary

$$\begin{cases} -\Delta u = f, \ in \ \Omega, \\ u = 0, \ on \ \partial\Omega. \end{cases}$$

For give integer $k \geq 1$, on each element T, define local weak finite element space

$$S(k, T) = \{v = \{v_0, v_b\} : \ v_0 \in P_k(T), \ v_b|_e \in P_k(e)\},$$

and the global weak finite element spaces S_h and S_h^0 as

$$S_h = \left\{v = \{v_0, v_b\} : \ v|_T \in S(k, T), \ v_b|_{\partial T_1 \cap e} = v_b|_{\partial T_2 \cap e}, \ T \in \mathcal{T}_h, \ e \in \mathcal{E}_h^0\right\},$$
$$S_h^0 = \{v \in S_h : \ v_b|_{\partial\Omega} = 0\}.$$

The weak Galerkin formulation of this Poisson equation is: find $u_h \in S_h^0$ such that

$$(\nabla_w u_h, \nabla_w v_h) + s(u_h, v_h) = (f, v_0), \ \forall v_h = \{v_0, v_b\} \in S_h^0.$$

Here, $s(u_h, v_h)$ is the stabilization term to ensure the weak continuity of u_h. Please identify the formulation of $s(u_h, v_h)$ for this finite element space. You may find the L^2 projection operators are useful. Please specify the projections you use and figure out the stabilization term.

Exercise 31. For any element $T \in \mathcal{T}_h$, assume that Q_0 is the L^2 projection from $L^2(T)$ to $P_k(T)$, and thatQ_b is the L^2 projection from $L^2(T)$ to $P_{k-1}(e)$. We define projection operator $Q_h : H^1(\Omega) \to S_h$ such that on each element

$$Q_h v = \{Q_0 v_0, Q_b v_b\}.$$

And $\mathbb{Q}_h$ is the L^2 projection to $[P_{k-1}(T)]^d$. Prove on a given element T,

$$\nabla_w(Q_h \phi) = \mathbb{Q}_h(\nabla\phi), \ \forall \phi \in H^1(T).$$

Exercise 32. Assume that $\mathcal{T}_h$ is a triangulation of the domain Ω, and $K \in \mathcal{T}_h$ is an arbitrary element. Denote by $\mathcal{E}_h$ the union of the boundaries of the elements K of $\mathcal{T}_h$, $\mathcal{E}_h^i = \mathcal{E}_h \backslash \partial\Omega$ is the set of interior edges and $\mathcal{E}_h^\partial = \mathcal{E}_h \backslash \mathcal{E}_h^i$ is the set of boundary edges. Let e be the common edge of two elements K^+ and K^-, and $n^i = n|_{\partial K^i}$ be the unit outward normal vector on ∂K^i with $i = +, -$. For $v \in T(\mathcal{E}_h)$, let $v^i = v|_{\partial K^i}$, and similarly, for $\boldsymbol{q} \in [T(\mathcal{E}_h)]^2$, we denote $\boldsymbol{q}^i = \boldsymbol{q}|_{\partial K^i}$. Then we define the average $\{\cdot\}$ and the jump $[\cdot]$ on $e \in \mathcal{E}_h^i$ by

$$\{v\} = \frac{1}{2}(v^+ + v^-), \qquad [v] = v^+ n^+ + v^- n^-,$$
$$\{\boldsymbol{q}\} = \frac{1}{2}(\boldsymbol{q}^+ + \boldsymbol{q}^-), \qquad [\boldsymbol{q}] = \boldsymbol{q}^+ \cdot n^+ + \boldsymbol{q}^- \cdot n^-.$$

If $e \in \mathcal{E}_h^\partial$, we set

$$[v] = vn, \quad \{\boldsymbol{q}\} = \boldsymbol{q} \quad \text{on } e \in \mathcal{E}_h^\partial,$$

where n is the outward unit normal.

For a scalar-valued function v and a vector-valued function $\boldsymbol{w}$, after a direct manipulation, we have

$$\sum_{K\in\mathcal{T}_h} \int_{\partial K} (v\boldsymbol{n}_K) \cdot \boldsymbol{w}\, ds = \sum_{e\in\mathcal{E}_h} \int_e [v] \cdot \{\boldsymbol{w}\}\, ds + \sum_{e\in\mathcal{E}_h^i} \int_e \{v\}[\boldsymbol{w}]\, ds. \tag{23.62}$$

Consider the Poisson equation with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, \ in\ \Omega, \\ u = g, \ on\ \partial\Omega. \end{cases}$$

On each element $K \in \mathcal{T}_h$, $u \in H^2(\Omega)$ satisfies the following identity

$$\int_K \nabla u \cdot \nabla v\, dx - \int_{\partial K} (\nabla u \cdot \boldsymbol{n}) v\, ds + \int_{\partial K} \mu(u - u^{nb}) v\, ds = \int_K f\, v\, dx$$

for all the $v \in H^2(\mathcal{T}_h)$. Here, u^{nb} is the value of the u on the edge of the neighbor element, and equal g on the boundary. Adding over all the elements $K \in \mathcal{T}_h$, we have

$$\int_\Omega \nabla u \cdot \nabla_h v\, dx - \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\nabla u \cdot \boldsymbol{n}) v\, ds + \sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu(u - u^{nb}) v\, ds = \int_\Omega f\, v\, dx \tag{23.63}$$

(a) Please prove the identity (23.62).

(b) Given a function space $V_h = \left\{v \in L^2 : \ v\,|_K \in \mathcal{P}_r(K), \ \forall K \in \mathcal{T}_h\right\}$, by the help of identity (23.62) and (23.63), please derive the following Interior Penalty (IP) formulation for this Poisson equation in detail: find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = \int_\Omega f v_h + \int_{\partial\Omega} \mu g v_h - \int_{\partial\Omega} g \frac{\partial v_h}{\partial \boldsymbol{n}}, \ \forall v_h \in V_h,$$

where $\boldsymbol{n}$ is the outer normal vector of $\partial\Omega$. And $a_h(u_h, v_h)$ is given by

$$a_h(u_h, v_h) = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] - \int_{\mathcal{E}_h} [u_h]\{\nabla_h v_h\} + \int_{\mathcal{E}_h} \mu[u_h][v_h].$$

Usually, we let the penalty constant $\mu = \eta h^{-1}$.

Exercise 33. Define DG space

$$V_h = \left\{ v \in L^2 : \ v\,|_K \in \mathcal{P}_r(K), \ \forall K \in \mathcal{T}_h \right\}.$$

Prove that for any $v, w \in V_h + H^2(\mathcal{T}_h) \cap H_0^1(\Omega)$,

$$a_h(v, w) \le C\|v\|_{DG}\|w\|_{DG},$$

where C is a constant independent of h. And the DG norm is defined as and DG norm

$$\|v\|_{DG}^2 = \sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 \, dx + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \sum_{e\in\mathcal{E}_h} \int_e h^{-1} |[v]|^2 \, ds.$$

Exercise 34. Define bilinear form

$$a_h(u_h, v_h) = \int_\Omega \nabla_h u_h \cdot \nabla_h v_h - \int_{\mathcal{E}_h} \{\nabla_h u_h\} \cdot [v_h] - \int_{\mathcal{E}_h} [u_h]\{\nabla_h v_h\} + \int_{\mathcal{E}_h} \frac{\eta}{h}[u_h][v_h],$$

and DG norm

$$\|v\|_{DG}^2 = \sum_{K\in\mathcal{T}_h} \int_K |\nabla v|^2 \, dx + \sum_{K\in\mathcal{T}_h} h_K^2 |v|_{2,K}^2 + \sum_{e\in\mathcal{E}_h} \int_e h^{-1} |[v]|^2 \, ds.$$

Prove that for sufficiently large $\eta > 0$,

$$a_h(v_h, v_h) \ge \alpha \|v_h\|_{DG}^2, \ \forall v_h \in V_h.$$

And α is a positive constant independent of h.

23.15 Multigrid

Exercise 35. Let Q_h be the L^2 projection to piecewise linear continuous space V_h. Prove that

$$\|Q_h u\|_\alpha \lesssim \|u\|_\alpha, \quad \forall u \in H_0^\alpha(\Omega), \quad 0 \le \alpha \le 1.$$

Exercise 36. Given $\gamma \in (0, 1)$, then

$$\sum_{i,j}^\infty \gamma^{|i-j|} x_i x_j \le \frac{2}{1-\gamma} \sum_{i=1}^\infty x_i^2.$$

23.16 Conjugate gradient method

Exercise 37. Show with the help of an example that if the step length α is badly chosen, the gradient descent might diverge.

Hint: The iteration for gradient descent is given by $x_{k+1} = x_k - \alpha\nabla f(x_k)$. Consider the 1D case

Exercise 38. Verify that the functions $T_k(t)$ given by (**??**) are indeed polynomials of degree k. *hint:* Use the formula

$$\cos(k+1)x = -\cos(k-1)x + 2\cos kx \cos x$$

and similar formula for $\cosh(k+1)x$ to establish a recurrence relation between $T_k(t)$.

Exercise 39. The conjugate-gradient method is applied to the minimization of a function. At some iteration the following data were given: $r_i = (5, 3, -1)^T$ and $p_i = (4, -2, 1)^T$. Why cannot these data be correct?

Exercise 40. Show that the conjugate-gradient method can be adjusted to take adavantage of some initial guess $x_0 \neq 0$ by taking $r_0 = b - Ax_0$.

Hint: Show that this is equivalent to applying the method to a related linear system with initial guess equal to zero.

Exercise 41. Assume we have a diagonal matrix A of order of N = 100, the eigenvalues of which are in the range of $[10^{-p}, 1]$, for p = 1, 2, 3, respectively. Since all eigenvalues of A are positive, A is SPD.

a) For each p, construct the diagonal matrix A with eigenvalues equally distributed in the range $[10^{-p}, 1]$, i.e., $\lambda_i = 10^{-p} + \frac{i(1-10^{-p})}{N-1}$, i = 0, 1, . . . , N-1. Use both steepest descent and CG method (without preconditioning) to solve Ax = b with b = rand(N,1) and the starting vector x0 = rand(N,1). Build a table to compare the iteration steps or plot the residuals for both methods (Keep other parameters such as tolerance the same). Which method converges faster?

b) Now let us consider another distribution of eigenvalues in the range$[10^{-p}, 1]$, i.e., $\lambda_i = 10^{-p} + (1 - 10^{-p})\cos\frac{i\pi}{2(N-1)}$, i = 0, 1, . . . , N-1. Then follow the same procedure as in (a). Compare the iteration steps of (a) and (b). Which method converges faster? Can you explain?

(Hint: In (b), the eigenvalues are clustered close to 1.)

Exercise 42. If $B \in \mathbb{R}^{n\times n}$ is a symmetric and positive definite matrix, by choosing B as an approximation to A^{-1}, $x \to Bx$ can be computed fast. Therefore we would have an equivalent linear system $BAx = Bb$. Aim of preconditioning is to require that $cond_2(BA) \ll cond_2(A)$, in order to have a faster convergence of the CG method.

a) Let $B \in \mathbb{R}^{n\times n}$ be a given symmetric and positive definite matrix and consider the problem $B^{\frac{1}{2}}AB^{\frac{1}{2}}y = B^{\frac{1}{2}}b$. Derive the CG-algorithm to solve $Ax = b$, but reformulate it in order to obtain an algorithm where the full matrix $B^{\frac{1}{2}}AB^{\frac{1}{2}}$ is not used.

b) Let us consider the following preconditioners:
   1) Jacobi method: $B_J = D$;
   2) Symmetric Gauss-Seidel method: $B_{SGS} = (D + L)\cdot(D^{-1}(D + U))$;
   3) Symmetric SOR method: $B_{SSOR} = \frac{1}{2-\omega}((\frac{1}{-\omega} + L)\cdot(\frac{1}{\omega}D)^{-1}(\frac{1}{\omega}D + U))$ for $\omega = 1.5$.

Using Matlab's function pcg, test the CG-solver with and without the given preconditioners for the Poisson matrix

$$A = diag(-I_d, B, I_d) \in \mathbb{R}^{N^2\times N^2},\ B = diag(-1, 4, -1) \in \mathbb{R}^{N\times N},$$

with $I_d \in \mathbb{R}^{N\times N}$ and the right-hand side $b = (1\cdots 1) \in \mathbb{R}^{N^2}$ for N = 1, . . . , 30. Plot the numbers of unknown vs. the number of iterations.

1. Consider solving the linear system of equations

$$\mathsf{A}\mathsf{u} = \mathsf{f},$$

by the method of subspace correction. Here $\mathsf{A} \in \mathbb{R}^{n\times n} : \mathbb{R}^n \mapsto \mathbb{R}^n$ is symmetric positive definite.
 a) Consider the simple decomposition of $\mathbb{R}^n$

$$\mathbb{R}^n = \sum_{i=1}^{n} \operatorname{span}\{e_i\}, \tag{23.64}$$

 where e_i is the i-th column of the identity matrix. Show that the parallel subspace correction method (with exact subspace solver) is just the Jacobi iterative method.
 b) Consider the same decomposition (23.64), show that the successive subspace correction method (with exact subspace solver) is just the Gauss-Seidel iterative method.
2. Prove that Jacobi method converges for any 2×2 symmetric positive definite matrices.
3. Consider using subspace correction method to solve $Au = f$,
 a) Show that the parallel subspace correction method for $Au = f$ is equivalent to the modified Jacobi method for $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:

$$\underset{\sim}{u}^m = \underset{\sim}{u}^{m-1} + \underset{\sim}{R}(\underset{\sim}{f} - \underset{\sim}{A}\underset{\sim}{u}^{m-1})$$

 b) Show that the successive subspace correction method for $Au = f$ is equivalent to the modified G-S method for $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$:

$$\underset{\sim}{u}^m = \underset{\sim}{u}^{m-1} + (\underset{\sim}{R}^{-1} + \underset{\sim}{L})^{-1}(\underset{\sim}{f} - \underset{\sim}{A}\underset{\sim}{u}^{m-1}).$$

 Define $\underset{\sim}{B} = (\underset{\sim}{R}^{-1} + \underset{\sim}{L})^{-1}$, prove that $B = \Pi \underset{\sim}{B} \Pi'$, where B is the preconditioner of the original SSC iteration

$$u^m = u^{m-1} + B(f - Au^{m-1}).$$

4. Consider the linear system $Au = f$. Here $A : \mathbb{R}^{n\times n} \to \mathbb{R}^n$ is an SPD matrix. Consider the following iterative method

$$u^m = u^{m-1} + B(f - Au^{m-1}), \tag{23.65}$$

 where B is an arbitrary matrix. And we have a symmetrized iterative method,

$$u^m = u^{m-1} + \bar{B}(f - Au^{m-1}), \tag{23.66}$$

 where $\bar{B} = B^T + B - B^T AB$. Construct an example such that the iteration (23.65) converges, while (23.66) does not.
5. Assume that H is a Hilbert space, and $H_i \subset H$ $(i = 1, \cdots, m)$ is a sequence of subspaces. Assume that $P_{H_i} : H \to H_i$ are the projections defined by

$$(P_{H_i}u, v) = (u, v_i), \ \forall v_i \in H_i, \ i = 1, \cdots, m.$$

 Prove that

$$\lim_{k\to\infty} (P_{H_1} \cdots P_{H_m})^k = P_{H_1\cap H_2\cap\cdots\cap H_m},$$

 where $P_{H_1\cap H_2\cap\cdots\cap H_m}$ is the projection from H to $H_1 \cap H_2 \cap \cdots \cap H_m$.

6. Consider the Poisson equation

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \partial\Omega \end{cases}$$

discretized by linear finite element method on a uniform triangulation grid (see Figure 6) of $\Omega = (0,1)\times(0,1)$ with mesh size h. Here, we have

$$0 = x_0 < x_1 < \cdots < x_{n+1} = 1, \quad x_j = \frac{j}{n+1}, \quad (j = 0, \cdots, n+1).$$

and

$$0 = y_0 < y_1 < \cdots < y_{n+1} = 1, \quad y_j = \frac{j}{n+1}, \quad (j = 0, \cdots, n+1).$$

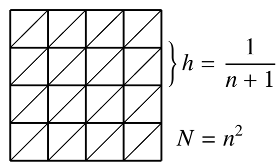

Fig. 23.12. Two-dimensional uniform grid for finite element method

The weak form is: Find $u_h \in V_h$, such that

$$a(u_h, v_h) = (f, v_h), \ \forall\, v_h \in V_h \tag{23.67}$$

where $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h$ and $(f, v_h) = \int_\Omega f v_h$, and V_h is the corresponding linear finite element space. Take $v_h = \phi^h_{i,j}$ where $\phi^h_{i,j}$ is the basis function at point (x_i, y_j), from (23.67), we have

$$4\mathsf{u}_{i,j} - \mathsf{u}_{i-1,j} - \mathsf{u}_{i+1,j} - \mathsf{u}_{i,j-1} - \mathsf{u}_{i,j+1} = \mathsf{f}_{i,j},$$

where $\mathsf{f}_{i,j} = (f, \phi^h_{i,j})$. This leads to the following linear system of equations.

$$\mathsf{A}\mathsf{u} = \mathsf{f}.$$

where $\mathsf{A} = \text{tridiag}(-\mathsf{I}, \mathsf{B}, -\mathsf{I})$ and $\mathsf{B} = \text{tridiag}(-1, 4, -1)$ and $\mathsf{u} = (\mathsf{u}_{i,j})$ and $\mathsf{f} = (\mathsf{f}_{i,j})$ with i and j both follow the lexicographic ordering.
Implement the following Gauss-Seidel method:

Gauss-Seidel method: $[\mathsf{u}] = \mathsf{GS}(\mathsf{u}, \mathsf{f}, n)$

```
For j = 1 : n
    For i = 1 : n
        u_{i,j} ← (f_{i,j} + u_{i-1,j} + u_{i+1,j} + u_{i,j-1} + u_{i,j+1})/4
    end For
end For
```

Given an initial guess u^0, consider the Gauss-Seidel iteration as follows:

$k \leftarrow 0$
While $\|\mathsf{f} - \mathsf{A}\mathsf{u}^k\| / \|\mathsf{f} - \mathsf{A}\mathsf{u}^0\| > \mathtt{Tol}$
  $[\mathsf{u}^{k+1}] = \mathsf{GS}(\mathsf{u}^k, \mathsf{f}, n)$
  $k \leftarrow k + 1$
end While

a) Choose right hand side f and initial guess u^0 freely (constant right hand side and random initial guess are recommended), and set the tolerance to be $\mathtt{Tol} = 10^{-6}$. Make a table to report the number of iterations, convergence factor ($\|\mathsf{f} - \mathsf{A}\mathsf{u}^k\| / \|\mathsf{f} - \mathsf{A}\mathsf{u}^{k-1}\|$), and CPU time for $h = 2^{-3}, 2^{-4}, 2^{-5}, 2^{-6}$.

b) $\mathcal{T}_i$ ($i = 1, \cdots, J$) is a sequence of meshes formed by uniform refinement (figure 23.15), with mesh size h_i. Assume that $\left\{\phi_k^{(i)}\right\}_{k=1}^{n_i}$ is the set of nodal basis function on $\mathcal{T}_i$ ($i = 1, \cdots, J$). The expanded

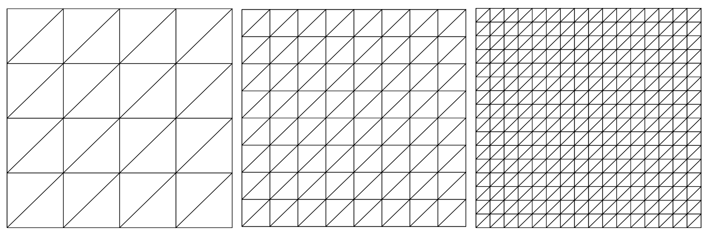

Fig. 23.13. A sequence of uniformly refined mesh

system $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$ can be formed by

$$\underset{\sim}{A} = \left((\phi_i^{(l)}, \phi_j^{(k)})\right),\ i = 1, \cdots, n_l,\ j = 1, \cdots, n_k,\ l, k = 1, \cdots, J,$$
$$\underset{\sim}{f}_j^{(l)} = (f, \phi_i^{(l)}),\ i = 1, \cdots, n_l,\ l = 1, \cdots, J.$$

Choose $h_1 = 2^{-3}$ (and $h_{i+1} = h_i/2$), $J = 4$, form the expanded system $\underset{\sim}{A}\underset{\sim}{u} = \underset{\sim}{f}$ and use Gauss-Seidel method to solve this expanded system.

23.17 Nonconforming finite element method

Exercise 43. Consider the Poisson problem with Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

The variational form of this problem is: find $u \in H_0^1$ s.t.

$$a(u, v) = \langle u, v \rangle,\ \forall v \in H_0^1,$$

where $a(u, v) = (\nabla u, \nabla v)_{L^2}$. Define space

$$V_h^{CR} = \{v :\ v|_K \in P_1(K),\ v \text{ is continuous at the midpoint of each edge}\}.$$

The non-conforming finite element discretization is: find $u_h \in V_h^{CR}$ s.t.

$$a_h(u_h, v_h) = (f, v_h)_{L^2},\ \forall v_h \in V_h^{CR},$$

where $a_h(u_h, v_h) = \sum\limits_K (\nabla u_u, \nabla v_h)_{L^2}$. Prove the following error estimate holds

$$|u - u_h|_{1,h} := \left(\sum_K |u - u_h|_{1,K}^2\right)^{1/2} \leq Ch,$$

where C is a constant independent of h.

Exercise 44. Define space

$$V_h^{CR} = \{v :\ v\,|_K \in P_1(K),\ v \text{ is continuous at the midpoint of each edge}\}.$$

1. Find the nodal basis functions $\{\psi_1, \cdots, \psi_n\}$ of V_h^{CR}.
2. Prove that $\int_\Omega \psi_i \psi_j dx = 0$, for any $i \neq j$.

23.18 Take-home exams

Due to October 21, 2018. Please try to do all the problems

1. Given

$$A_0 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix} \in R(A_0)$$

a) Prove that A_0 is positive semi-definite and find all the eigenvalues and eigenvectors of A_0.
b) Prove that $(A_0 + \epsilon I)x = b$ is uniquely solvable for $\epsilon > 0$ and also solvable (and unique up to a constant vector) for $\epsilon = 0$.
c) Apply the Gauss-Seidel method for $(A_0 + \varepsilon I)x = b$

$$\begin{cases} a_{11}x_1^m + a_{12}x_2^{m-1} + a_{13}x_3^{m-1} = b_1 \\ a_{21}x_1^m + a_{22}x_2^m + a_{23}x_3^{m-1} = b_2 \\ a_{31}x_1^m + a_{32}x_2^m + a_{33}x_3^m = b_3 \end{cases}$$

Using the initial guess $x^0 = b$, record the minimal number of iteration m satisftying the stopping criterion that $\|Ax^m - b\| \leq 10^{-6}$:

ε	# of iter = m
1.	
10^{-1}	
10^{-2}	
10^{-3}	
10^{-4}	
10^{-5}	
10^{-6}	
10^{-7}	
10^{-8}	
10^{-9}	
0. [singular case]	

d) Provide a theoretical analysis of $\|x - x^m\|_A$ for small $\epsilon > 0$ and $\epsilon = 0$, respectively. Use this theoretical result to justify the numerical results you got.

2. Assume that

$$K = \text{tetrahedron} = [a_0, a_1, a_2, a_3] \subseteq \mathbb{R}^2,$$
$$\mathcal{P} = P_2(K) = \{p(x) : \ p(x) \text{ is a polynomial on K and its order} \leq 2\},$$
$$\mathcal{N}(v) = \{v(a_1) : \ 0 \leq i \leq 3\} \cup \left\{v(a_{ij}) : \ 0 \leq i < j \leq 3\right\}, \ a_{ij} = \frac{1}{2}(a_i + a_j).$$

Prove that the triple $(K, \mathcal{P}, \mathcal{N})$ is a well-defined finite element space.

3. Assume that τ is a d-simplex in $\mathbb{R}^d$ ($d = 2$ or 3). Namely, τ is a triangle in $\mathbb{R}^2$, and a tetrahedron in $\mathbb{R}^3$. λ_i ($i = 1, \cdots, d+1$) is the barycentric coordinate at vertex i. Prove that

$$\int_\tau \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \cdots \lambda_{d+1}^{\alpha_{d+1}} dx = |\tau| \frac{d!\alpha_1!\alpha_2! \cdots \alpha_{d+1}!}{(d + \alpha_1 + \alpha_2 + \cdots + \alpha_{d+1})!}.$$

4. Define space

$$V_h^{CR} = \{v : \ v\,|_K \in P_1(K), \ v \text{ is continuous at the midpoint of each edge}\}.$$

a) On each element K, find the nodal basis in terms of barycentric coordinates λ_1, λ_2 and λ_3.

b) Find the nodal basis functions $\{\psi_1, \cdots, \psi_n\}$ of V_h^{CR} and prove that $\int_\Omega \psi_i \psi_j dx = 0$, for any $i \neq j$.

c) Let $Q_h : L^2(\Omega) \mapsto V_h^{CR}$ be the L^2 projection. Given any $f \in L^2(\Omega)$, find explicit expression of $\alpha_i(f)$ such that

$$(Q_h f)(x) = \sum_{i=1}^n \alpha_i(f) \psi_i(x)$$

d) Prove that, for any $u \in H^1(\Omega)$

$$\|u - Q_h u\|_{L^2(\Omega)} = \inf_{v_h \in V_h^{CR}} \|u - v_h\|_{L^2(\Omega)} \leq Ch|u|_{1,\Omega}.$$

5. For any smooth function $v \in C^\infty$, the interpolation operators are defined as

- $\Pi_h v = \sum\limits_{i=1}^{N} v(a_i)\phi_i^{(0)}$, where N is the total number of vertices in the mesh, a_i are the vertices.
- $\Pi_h^{\text{curl}} v = \sum\limits_{i=1}^{NE} \left(\int_{e_i} v \cdot \tau_{e_i} dl\right) \phi_i^{(1)}$, where NE is the total number of edges in the mesh, e_i is the ith edge, τ_{e_i} is the unit tangential vector along the edge.
- $\Pi_h^{\text{div}} v = \sum\limits_{i=1}^{NF} \left(\int_{F_i} v \cdot n_{F_i} ds\right) \phi_i^{(2)}$, where NF is the total number of faces in the mesh, F_i is the ith face, n_{F_i} is the unit outer normal vector of the face F_i.
- $\Pi_h^0 v = \sum\limits_{i=1}^{NT} \left(\int_{T_i} v dx\right) \phi_i^{(3)}$, where NT is the total number of elements (tetrahedrons) in the mesh, T_i is the ith element.

Prove that all the diagrams shown in Fig. 5 are commutative, namely

a) $\text{grad}\Pi_h v = \Pi_h^{\text{curl}} \text{grad} v$.

b) $\text{curl}\Pi_h^{\text{curl}} v = \Pi_h^{\text{div}} \text{curl} v$.

c) $\text{div}\Pi_h^{\text{div}} v = \Pi_h^0 \text{div} v$.

6. Based on the nested sequence of grids as showing in Fig. 23.15, we have a nested sequence of linear finite spaces

$$V_1 \subset V_2 \subset \cdots V_J = V_h,$$

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & C^\infty & \xrightarrow{\text{grad}} & C^\infty & \xrightarrow{\text{curl}} & C^\infty & \xrightarrow{\text{div}} & C^\infty & \longrightarrow & 0 \\
 & & \big\downarrow \Pi_h & & \big\downarrow \Pi_h^{\text{curl}} & & \big\downarrow \Pi_h^{\text{div}} & & \big\downarrow \Pi_h^0 & & \\
R & \longrightarrow & H_h^1 & \xrightarrow{\text{grad}} & H_h^{\text{curl}} & \xrightarrow{\text{curl}} & H_h^{\text{div}} & \xrightarrow{\text{div}} & L_h^2 & \longrightarrow & 0
\end{array}
$$

Fig. 23.14. Exact sequences and commutative diagrams

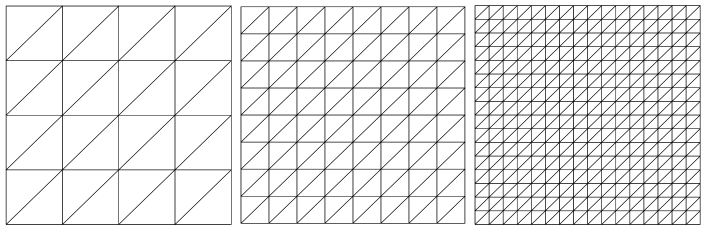

Fig. 23.15. A sequence of uniformly refined mesh

where $V_k = \left\{\phi_i^k\right\}_{i=1}^{n_k}$ with $\left\{\phi_i^k\right\}_{i=1}^{n_k}$ being the set of nodal basis function on mesh $\mathcal{T}_k$ $(k = 1, \cdots, J)$ Assume that $R_j : V_j' \to V_j$ represents the Jacobi method and i_k is the inclusion operator from V_k to V_h. The operator form of PSC is given by

$$B_h = \sum_{k=1}^{J} i_k R_k i_k'.$$

a) For any $f_h \in V_h'$, compute $B_h f_h$ in terms of nodal basis functions on all levels

$$\left\{\phi_i^k : i = 1 : n_k, k = 1 : J\right\}.$$

b) Prove that the matrix representation of B_h is given by

$$\tilde{B} = \sum_{k=1}^{J} I_k D_k^{-1} I_k^T,$$

where D_k is the diagonal matrix of the stiffness matrix on V_k, I_k is the restriction matrix between V_k and V_h. That is, the (i, j)-th element of I_k is given by $\phi_j^k(x_i)$, where x_i is the ith node on mesh $\mathcal{T}_J$.

7. We consider the following Stokes problem on the computational domain $[0, 1] \times [0, 1]$:

$$
\begin{cases}
-\Delta \mathbf{u} + \nabla p = \mathbf{f} \text{ in } \Omega, \\
\text{div}\mathbf{u} = 0 \text{ in } \Omega, \\
\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega.
\end{cases} \tag{23.68}
$$

a) Define the $a(\cdot, \cdot)$ and $b(\cdot, \cdot)$ so that the variational formulation of the above Stokes equation is as follows: Find $\mathbf{u} \in V := (H_0^1(\Omega))^2$ and $p \in Q := L_0^2(\Omega)$ such that

$$
\begin{cases}
a(\mathbf{u}, \mathbf{v}) + b(v, p) = \langle \mathbf{f}, \mathbf{v} \rangle \ \forall\, \mathbf{v} \in V \\
b(\mathbf{u}, q) = 0 \quad \forall\, q \in Q.
\end{cases} \tag{23.69}
$$

b) Let $V_h \subset V$ consist of piecewise quadratic functions and $Q_h \subset Q$ consist of piecewise constant functions. Write out the corresponding finite element discretization for the Stokes equations

c) Write out the stiffness matrix of the finite element problem in the form of $\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}$, and write A and B in terms of certain basis functions for V_h and Q_h.

d) Prove that B has full rank.

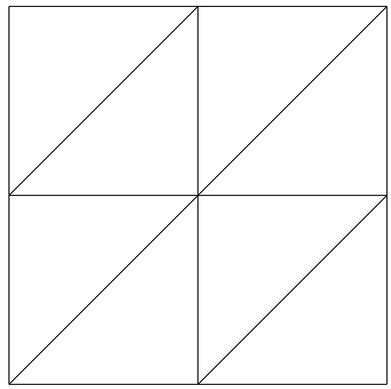

Fig. 23.16. The computational domain.

8. Assuem that $A \in \mathbb{R}^{n\times n}$ is SPD and $B \in \mathbb{R}^{n\times m}$ is of full-rank. Prove that the following "preconditioned matrix" (with $S = BA^{-1}B^T$):

$$\begin{pmatrix} A & 0 \\ 0 & S \end{pmatrix}^{-1} \begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix}$$

only has three distinctive eigenvalues: $1, (1 \pm \sqrt{5})/2$. Determine the multiplicity of these eigenvalues.

23.18.1 Final

Due time: 5pm, December 12, 2018

Instruction

1. You are required to work on all the problems.
2. No finite element software can be used to do the programming problem. Email me (via xu@math.psu.edu) all the source code with your computational results.
3. Online submission of your solutions via email xu@math.psu.edu is encouraged.
4. If it is not 100% online submission, please place your hand-written solutions to the exam problems under the door of my office: 314 McAllister Building

Problem: Consider the following 2nd order elliptic boundary value problem

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases} \tag{23.70}$$

The Hybridized DG formulation of problem (23.70) is: Find $(p_h, u_h, \hat{u}_h) \in Q_h \times V_h \times M_h$ such that for any $(q_h, v_h, \hat{v}_h) \in Q_h \times V_h \times M_h$,

$$\begin{cases} \sum_K (c_K p_h, q_h)_K + \sum_K (u_h, \nabla \cdot q_h)_K - \sum_K \langle \hat{u}_h, q_h \cdot n \rangle_{\partial K} = 0, \\ \sum_K (\nabla \cdot p_h, v_h)_K - \sum_K \tau \langle u_h, v_h \rangle_{\partial K} + \sum_K \tau \langle \hat{u}_h, v_h \rangle_{\partial K} = -\langle f, v_h \rangle, \\ -\sum_K \langle p_h \cdot n, \hat{v}_h \rangle_{\partial K} + \sum_K \tau \langle u_h, \hat{v}_h \rangle_{\partial K} - \sum_K \tau \langle \hat{u}_h, \hat{v}_h \rangle_{\partial K} = 0, \end{cases} \tag{23.71}$$

where $K \in \mathcal{T}_h$ is an arbitrary element and $c_K = (\alpha \mid_K)^{-1}$. The operator form of the left-hand side of the above system is:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ \mathcal{B} & -C \end{pmatrix}$$

$\mathcal{B}'$ is the adjoint operator of $\mathcal{B}$. Here,

$$\begin{aligned} \left(\mathcal{A}\begin{pmatrix} p_h \\ u_h \end{pmatrix}, \begin{pmatrix} q_h \\ v_h \end{pmatrix}\right) &= \sum_K \left(\mathcal{A}_K \begin{pmatrix} p_h \\ u_h \end{pmatrix}, \begin{pmatrix} q_h \\ v_h \end{pmatrix}\right)_K, \\ \left(\mathcal{B}\begin{pmatrix} p_h \\ u_h \end{pmatrix}, \hat{v}_h\right) &= \sum_K \left(\mathcal{B}_K \begin{pmatrix} p_h \\ u_h \end{pmatrix}, \hat{v}_h\right)_K. \end{aligned}$$

And

$$\begin{aligned} \left(\mathcal{A}_K \begin{pmatrix} p_h \\ u_h \end{pmatrix}, \begin{pmatrix} q_h \\ v_h \end{pmatrix}\right)_K &= \begin{pmatrix} (c_K p_h, q_h)_K + (u_h, \nabla \cdot q_h)_K \\ (\nabla \cdot p_h, v_h)_K - \tau \langle u_h, v_h \rangle_{\partial K} \end{pmatrix}, \\ \left(\mathcal{B}_K \begin{pmatrix} p_h \\ u_h \end{pmatrix}, \hat{v}_h\right)_K &= -\langle p_h \cdot n, \hat{v}_h \rangle_{\partial K} + \tau \langle u_h, \hat{v}_h \rangle_{\partial K}. \end{aligned}$$

Define finite element space

$$\begin{aligned} Q_h &= \left\{ q \in \left[L^2(\Omega)\right]^2 : \ q \mid_K \in (\mathcal{P}_0)^2 + \mathcal{P}_0 \mathbf{x} \right\}, \\ V_h &= \left\{ v \in L^2(\Omega) : \ v \mid_K \in \mathcal{P}_0 \right\}, \\ M_h &= \left\{ \hat{v} \in L^2(\mathcal{E}_h) : \ v \mid_e \in \mathcal{P}_0 \text{ and } v \mid_{\mathcal{E}_h^\partial = 0} \right\}. \end{aligned}$$

Here, $\mathcal{P}_0$ is constant space. $\mathcal{E}_h$ is the union of all edges in the mesh $\mathcal{T}_h$ and $\mathcal{E}_h^\partial = \mathcal{E}_h \cap \partial\Omega$.

1. We can discretize problem (23.70) by mixed finite element method with the lowest order Raviart-Thomas element. That is, find $(p_h, u_h) \in Q_h^{RT} \times V_h$, such that for any $(q_h, v_h) \in Q_h^{RT} \times V_h$,

$$\begin{cases} (cp_h, q_h) + (u_h, \nabla \cdot q_h) = 0, \\ -(\nabla \cdot p_h, v_h) = \langle f, v_h \rangle. \end{cases} \tag{23.72}$$

Here the Q_h^{RT} is defined as

$$Q_h^{RT} = \left\{ q \in \left[L^2(\Omega)\right]^2 : \ q \mid_K \in (\mathcal{P}_0)^2 + \mathcal{P}_0 \mathbf{x}, \int_e [q] = 0, \forall e \in \mathcal{E}_h^i \right\},$$

where $\mathcal{E}_h^i = \mathcal{E}_h \setminus \mathcal{E}_h^\partial$ is the set of all inner edges in the mesh $\mathcal{T}_h$.

a) Verify that the following are local basis functions of Q_h^{RT} and V_h in element $K = [\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3]$ respectively

$$\begin{aligned} \phi_i &= \frac{\mathbf{x} - \mathbf{a}_i}{2|K|}, \quad \text{where the D.O.Fs are } N_e^{RT}(q) = \int_e q \cdot n, \quad \forall e \in \partial K. \\ \varphi_i &= 1, \quad \text{where the D.O.F is } N_K^V(u) = \frac{1}{|K|} \int_K v. \end{aligned} \tag{23.73}$$

b) Prove that the solution (p_h, u_h) to this problem (23.72) is the same with the solution (p_h, u_h) to problem (23.71) when $\tau = 0$.

2. We also can discretize problem (23.70) by non-conforming finite element method with CR element, which is defined as

$$V_h^{CR} = \left\{ v_h \in L^2(\Omega):\ v_h|_K \in \mathcal{P}_1(K),\ \int_e [v_h] = 0,\ \forall e \in \mathcal{E}_h^i,\ \text{and } v_h = 0 \text{ at midpoints of } e \in \mathcal{E}_h^\partial \right\}.$$

The non-conforming finite element discretization of problem (23.70) is: find $u_h \in V_h^{CR}$ such that for any $v_h \in V_h^{CR}$,

$$\sum_K \int_K \alpha \nabla u_h \cdot \nabla v_h = \langle f, v_h \rangle. \tag{23.74}$$

   a) Verify that the local basis functions of V_h^{CR} corresponding to the D.O.F $N_e^{CR}(u) = \frac{1}{|e|}\int_e u$ in element K are $1 - 2\lambda_i$, where λ_i are the barycentric coordinates in K. Then, the local basis functions of M_h corresponding to $N_e^{CR}(\cdot)$ are

   $$\hat{\phi}_e = \begin{cases} 1 & \text{on } e, \\ 0 & \text{otherwise.} \end{cases}$$

   The one-to-one correspondence between the basis functions of V_h^{CR} and M_h can be given by the D.O.Fs $N_e^{CR}(\cdot)$.
   b) For piecewise constant α, prove that the stiffness matrix of problem (23.74) is exactly the same with $C + BA^{-1}B^T$ of HDG when $\tau = 0$ under the one-to-one correspondence between V_h^{CR} and M_h above, where $C + BA^{-1}B^T$ is the matrix form of Schur complement $\mathcal{C} + \mathcal{B}\mathcal{A}^{-1}\mathcal{B}'$.
   c) Find out the exact relationship between the HDG when $\tau = 0$ and the CR elements.
3. For any $\tau \geq 0$, prove that the Schur complement $\mathcal{C}+\mathcal{B}\mathcal{A}^{-1}\mathcal{B}'$ or its matrix form $C+BA^{-1}B^T$ is symmetric positive definite.
4. Consider the case that $\Omega = [0,1] \times [0,1]$ and that $\alpha = 1$. This Ω is partitioned into two triangles by a line connecting $(0,0)$ and $(1,1)$ (see figure 23.17 for the mesh and the indices for of vertices, edges and elements).
   a) Compute the stiffness matrix and right hand side of HDG formulation (23.71) with $\alpha = 1$. *Hint*: On each element K, the basis functions of Q_h are given by

   $$\begin{pmatrix}1\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\end{pmatrix}, \quad \frac{\mathbf{x} - \mathbf{x}_0}{h},$$

   where $\mathbf{x}_0$ is right-angled vertex of element K. The basis functions of (p_h, u_h) on each element are suggested to be written together to make A block-diagonal.
   b) Compute the stiffness matrix for the RT mixed finite element formulation (23.72) by using the basis functions in (23.73).
   c) Compute the stiffness matrix for nonconforming CR element for (23.74).
5. For the uniform mesh of $\Omega = [0,1] \times [0,1]$ with mesh size h (figure 23.18), find the matrix form of the operator

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ \mathcal{B} & -\mathcal{C} \end{pmatrix}.$$

6. Implement this HDG discretization. The computation domain is $\Omega = [0,1] \times [0,1]$. Use uniform grid to partition Ω (figure 23.18). Verify the correctness of your code by convergence test. That is, choose the exact solution to be

$$u(x,y) = sin(\pi x) sin(\pi y),$$

and $\alpha = 1$. Compute the corresponding f and use it in your code. For $h = 1, 1/2, 1/4, 1/8, 1/16, 1/32$, evaluate $\|u_h - u\|_{L^2}$ and $\|p_h - p\|_{L^2}$. If the code is correct, you should be able to see the first order convergence when $\tau = 0$. That is to say, if the mesh size if halved, the error is also halved. Please report the numerical behavior when $\tau > 0$.

$\underset{\sim}{a}$

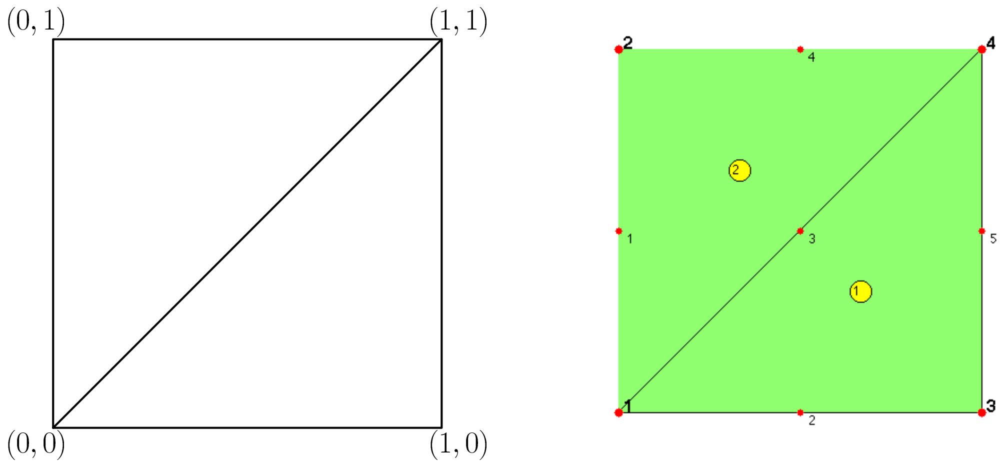

Fig. 23.17. A coarse mesh

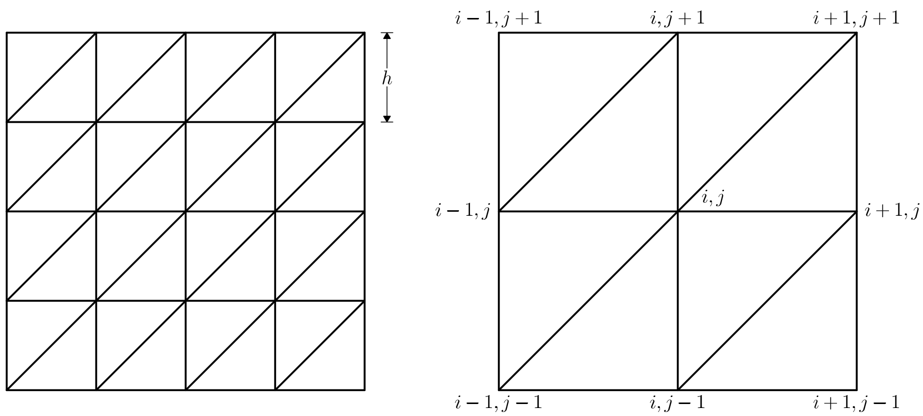

Fig. 23.18. Uniform mesh