10

Non-symmetric and/or indefinite problems

[c1] In this chapter, we give a brief discussion on a class of problems whose underlying PDE operators are not symmetric positive definite. Two classes of problems will be considered.

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We will mainly use some perturbation arguments to study these problems.

10.1 Some examples

10.1.1 General second elliptic boundary value problems

10.1.2 Second order problems with complex coefficients

We consider the following problems:

$$-\nabla \cdot (\alpha \nabla \mathbf{u}) = \mathbf{f}. \tag{10.1}$$

with boundary condition $u = 0$.

Let $\mathbf{V} = \mathbf{H}_0^1(\Omega)$ be the Sobolev spaces of complex functions. The variational formulation of (10.1) is: Find $\mathbf{u} \in V$

$$a(\mathbf{u}, \phi) \equiv (\alpha \nabla \mathbf{u}, \nabla \phi) = (\mathbf{f}, \phi) \quad \forall \phi \in \mathbf{V}. \tag{10.2}$$

Uniqueness

To study the well-posedness of the problem (10.4), let us first study the uniqness. We assume that $\mathbf{u} \in \mathbf{V}$ is such that:

$$a(\mathbf{u}, \phi) = 0 \quad \forall \phi \in \mathbf{V}. \tag{10.3}$$

We need to see under which conditions that only $\mathbf{u} = 0$ satisfies (10.3). To see this, let us first derive an equivalent formulation of (10.4).

We write:

$$\alpha = \alpha + i\beta, \mathbf{u} = u + iv, \mathbf{f} = f + ig.$$

We note that

$$\alpha \nabla \mathbf{u} = (a \nabla u - b \nabla v) + i(b \nabla u + a \nabla v).$$

It is easy to see that (10.4) is equivalent to: Find $\mathbf{u} \in V$ such that

$$\mathrm{Re}(\alpha \nabla \mathbf{u}, \nabla \phi) + \mathrm{Im}(\alpha \nabla \mathbf{u}, \nabla \psi) = (f, \phi) + (g, \psi), \quad \forall \phi, \psi \in V. \tag{10.4}$$

Let

$$\underset{\sim}{u} = (u, v), \underset{\sim}{\phi} = (\phi, \psi)$$

and

$$\underset{\sim}{a}(\underset{\sim}{u}, \underset{\sim}{\phi}) = \mathrm{Re}(\alpha\nabla \mathbf{u}, \nabla\phi) + \mathrm{Im}(\alpha\nabla \mathbf{u}, \nabla\psi) = (a\nabla u - b\nabla v, \nabla\phi) + (b\nabla u + a\nabla v, \nabla\psi).$$

Then

$$\underset{\sim}{a}((u, v), (u, v)) = (a\nabla u, \nabla u) + (a\nabla v, \nabla v)$$

and

$$\underset{\sim}{a}((u, v), (u, -v)) = (b\nabla u, \nabla u) + (b\nabla v, \nabla v).$$

Using the standard theory (such as Lax Milgram Lemma), we have

Lemma 69. *The variational problem* (10.4) *has at most one solution if one of the following conditions is satisfied:*

1. *There is a complex number β_0 and a positive constant c_0 satisfying:*

$$\mathrm{Re}(\beta_0\alpha(x)) \geq c_0, \quad \forall x \in \Omega.$$

2. *There are two constants α, β such that*

$$\alpha a(x) + \beta b(x) \geq \delta_0 > 0.$$

This condiiton implies

$$|a(w, \bar{\beta}_0 w)| \geq c_0 \|\nabla w\|^2, \quad \forall w \in \mathbf{V}.$$

10.1.3 A perturbation result

Define the bilinear form

$$\hat{a}(u, v) = (\nabla u, \nabla v).$$

Given any subspace $V_i \subset V$, we define $P_i, \hat{P}_i : V \mapsto V_i$ be the projections defined by

$$a(P_i u, v_i) = a(u, v_i), \quad \hat{a}(\hat{P}_i u, v_i) = a(u, v_i), \quad u \in V, v_i \in V_i.$$

It follows that

$$\begin{aligned}
\hat{a}((P_i - \hat{P}_i)u, v_i) &= \hat{a}(P_i u, v_i) - \hat{a}(\hat{P}_i u, v_i) \\
&= \hat{a}((P_i - I)u, v_i) - \alpha_i^{-1} a((P_i - I)u, v_i) \\
&= \alpha_i^{-1} \int_\Omega (\alpha_i - \alpha)(\nabla(P_i - I)u, \nabla v_i)
\end{aligned}$$

This implies that, if α is smooth and if $V_i \subset H_0^1(\Omega_i)$, then

$$|(P_i - \hat{P}_i)u|_{1,\Omega_i} \lesssim \mathrm{diam}(\Omega_i)|u|_{1,\Omega_i}.$$

10.2 Nonsymmetric and/or indefinite linear problems

In this section, we shall study a class of iterative methods for solving nonsymmetric or indefinite equations that are governed by some SPD systems. Straight iterative schemes as well as preconditioning techniques will be discussed.

This section is based on some early work by Xu and Cai [.xu cai.] and Xu [.xu-ncs.]. In [.xu cai.], a class of preconditioners are presented for GMRES type algorithms and in [.xu-ncs.] a class of linear iterative methods are developed. The algorithms in both [.xu cai.] and [.xu-ncs.] are built upon a small subspace solver and a given iterative method for the SPD operator that governs the equation, but the techniques used in these two papers are quite different. In this paper, we shall give a unified treatment for these algorithms, present some improved estimates and also propose some new preconditioners. We would like to mention that certain modifications for the algorithms in [.xu-ncs.] have been made by Bramble, Leyk and Pasciak [.blp.].

Model problems

In this section, we shall discuss finite element discretizations for nonsymmetric and/or indefinite linear partial differential equations. These results, mostly well-known, lay down the ground for the further analysis of nonlinear problems.

Linear elliptic partial differential operators

Let α, β, γ (with the ranges in $\mathbb{R}^{2\times 2}$, $\mathbb{R}^2$ and $\mathbb{R}^1$ respectively) be smooth functions on $\overline{\Omega}$ satisfying, for some positive constant α_0, that

$$\xi^T \alpha(x)\xi \geq \alpha_0 |\xi|^2 \quad \forall\, \xi \in \mathbb{R}^2.$$

We shall study the following two linear operators

$$\mathcal{L}\, v = -\mathrm{div}(\alpha(x)\nabla v) \quad \text{and} \quad \hat{\mathcal{L}}\, v = \mathcal{L}\, v + \beta(x)\cdot\nabla v + \gamma(x) v. \tag{10.1}$$

Obviously $\mathcal{L} : \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is an isomorphism. Our basic assumption is that $\hat{\mathcal{L}} : \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is also an isomorphism. (A simple sufficient condition for this assumption to be satisfied is that $\gamma(x) \geq 0$.) [c1]

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An application of the open mapping theorem yields

$$\|v\|_1 \lesssim \|\hat{\mathcal{L}} v\|_{-1} \quad \forall\, v \in \mathcal{H}_0^1(\Omega). \tag{10.2}$$

It is easy to see that if $\hat{\mathcal{L}}$ satisfies the above assumption, so does its formal adjoint:

$$\hat{\mathcal{L}}^* u = -\mathrm{div}(\alpha(x)\nabla u + \beta(x) u) + \gamma(x) u.$$

Namely $\hat{\mathcal{L}}^* : \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is also isomorphic and satisfies (10.2).

Corresponding to $\mathcal{L}$ and $\hat{\mathcal{L}}$, we define two bilinear forms, for $u, v \in \mathcal{H}_0^1(\Omega)$, as follows

$$A(u,v) = \int_\Omega \alpha(x)\nabla u\cdot\nabla v\, dx, \quad \hat{A}(u,v) = A(u,v) + \int_\Omega ((\beta\cdot\nabla u)v + \gamma(x) uv)\, dx. \tag{10.3}$$

We shall often use the following well-known regularity result (using (10.2)).

Lemma 70. *If $u \in \mathcal{H}_0^1(\Omega)$ and $\hat{\mathcal{L}} u \in \mathcal{L}^2(\Omega)$, then $u \in \mathcal{H}^2(\Omega)$ and*

$$\|u\|_2 \leq C\|\hat{\mathcal{L}} u\|$$

for some positive constant C depending on the coefficients of $\hat{\mathcal{L}}$ and the domain Ω.

Finite element discretizations

We assume that Ω is partitioned by a quasi-uniform triangulation $T_h = \{\tau_i\}$. By this we mean that τ_i's are simplexes of size h with $h \in (0,1)$ and $\bar{\Omega} = \cup_i \bar{\tau}_i$ and there exist constants C_0 and C_1 not depending on h such that each element τ_i is contained in (contains) a ball of radius $C_1 h$ (respectively $C_0 h$).

For a given triangulation T_h, a finite element space $\mathcal{V}_h \subset \mathcal{V} \equiv \mathcal{H}_0^1(\Omega)$ is defined by

$$\mathcal{V}_h = \{v \in C(\bar{\Omega}) : v|_\tau \in \mathcal{V}_\tau^r \quad \forall\, \tau \in T_h, v|_{\partial\Omega} = 0\},$$

where $\mathcal{V}_\tau^r$ is the space of polynomial of degree not greater than a positive integer r. For a given $v \in C(\bar{\Omega})$, $v_I \in \mathcal{V}_h$ will denote the standard nodal value interpolation of v.

It is well-known that (cf. [.ciarlet 1978.]) $\mathcal{V}_h$ satisfies the following approximation property

$$\inf_{\chi \in \mathcal{V}_h} \{\|v-\chi\|_{0,p} + h\|v-\chi\|_{1,p}\} \lesssim h^k |v|_{k,p}, \tag{10.4}$$

for all $v \in \mathcal{W}_p^k(\Omega) \cap \mathcal{H}_0^1(\Omega), 2 \le k \le r+1$ and $1 \le p \le \infty$.

Let $P_h : \mathcal{V} \longrightarrow \mathcal{V}_h$ be the standard Galerkin projection defined by

$$A(P_h v, \chi) = A(v, \chi) \quad \forall\, \chi \in \mathcal{V}_h. \tag{10.5}$$

Using Lemma 70 and a standard duality argument, we have

$$\|v - P_h v\| \lesssim h\|v\|_1 \quad \forall\, v \in \mathcal{V}. \tag{10.6}$$

For the nonsymmetric and/or indefinite problems, the following result (based on Schatz [.Schatz 1974.]), is of fundamental importance.

Lemma 71. *If $h \ll 1$, then*

$$\|v_h\|_1 \lesssim \sup_{\varphi \in \mathcal{V}_h} \frac{\hat{A}(v_h, \varphi)}{\|\varphi\|_1} \quad \text{and} \quad \|v_h\|_1 \lesssim \sup_{\varphi \in \mathcal{V}_h} \frac{\hat{A}(\varphi, v_h)}{\|\varphi\|_1} \quad \forall\, v_h \in \mathcal{V}_h. \tag{10.7}$$

The same results are also valid for $\varepsilon \ll 1$ if $\hat{A}$ in (10.7) is replaced by $\hat{A}_\varepsilon$ defined by

$$\hat{A}_\epsilon(u, v) = \int_\Omega (\alpha_\epsilon(x)\nabla u \cdot \nabla v + (\beta_\epsilon \cdot \nabla u)v + \gamma_\epsilon(x)uv)\, dx$$

with the functions $\alpha_\epsilon, \beta_\epsilon, \gamma_\epsilon \in L_\infty(\Omega)$ satisfying

$$\|\alpha - \alpha_\varepsilon\|_{0,\infty} + \|\beta - \beta_\varepsilon\|_{0,\infty} + \|\gamma - \gamma_\varepsilon\|_{0,\infty} = \delta_\varepsilon$$

where $\delta_\epsilon = o(1)$ as $\varepsilon \to 0$.

Proof. Since $\hat{\mathcal{L}} : \mathcal{H}_0^1(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is an isomorphism, we have

$$\|v_h\|_1 \lesssim \sup_{w \in \mathcal{V}} \frac{\hat{A}(v_h, w)}{\|w\|_1}.$$

Note that, by definition and (10.6)

$$\begin{aligned}
\hat{A}(v_h, P_h w) &= \hat{A}(v_h, w) - \hat{A}(v_h, w - P_h w) \\
&= \hat{A}(v_h, w) + (A - \hat{A})(v_h, w - P_h w) \\
&\ge \hat{A}(v_h, w) - c\|v_h\|_1 \|w - P_h w\| \\
&\ge \hat{A}(v_h, w) - c_1 h \|v_h\|_1 \|w\|_1.
\end{aligned}$$

The proof of the first estimate in (10.7) then follows by using the fact that $\|P_h w\|_1 \lesssim \|w\|_1$. The proof of the second estimate is similar.

For the form $\hat{A}_\epsilon(\cdot,\cdot)$, it follows from the assumption that

$$\hat{A}_\epsilon(v_h, \phi) \geq \hat{A}(v_h, \phi) - c\delta_\epsilon \|v_h\|_1 \|\phi\|_1.$$

The desired result then follows easily if $\epsilon \ll 1$. □

Now, define $\hat{P}_h : \mathcal{V} \longrightarrow \mathcal{V}_h$ by

$$\hat{A}(\hat{P}_h v, \chi) = \hat{A}(v, \chi) \quad \forall\, \chi \in \mathcal{V}_h. \tag{10.8}$$

Following (10.7) and Lemma 70, we have

Lemma 72. *If $h \ll 1$, then $\hat{P}_h$ is well-defined and*

$$\|u - \hat{P}_h u\| + h\|u - \hat{P}_h u\|_1 \lesssim h \inf_{\chi \in \mathcal{V}_h} \|u - \chi\|_1 \quad \forall\, u \in \mathcal{V}.$$

The following results show that P_h and $\hat{P}_h$ are "super-close" in $\mathcal{H}^1(\Omega)$ and $\mathcal{W}^1_\infty(\Omega)$ norms.

Lemma 73. *Assume that P_h and $\hat{P}_h$ are defined by* (10.5) *and* (10.8) *respectively, then*

$$\|P_h u - \hat{P}_h u\|_1 \lesssim \|u - \hat{P}_h u\|.$$

Proof. By definition

$$A(P_h u - \hat{P}_h u, \chi) = (A - \hat{A})(u - \hat{P}_h u, \chi) \quad \forall\, \chi \in \mathcal{V}_h.$$

The desired estimates then follow by taking $\chi = P_h u - \hat{P}_h u$. □

We end this section by stating some basic error estimates for $\hat{P}_h$.

Lemma 74. *The projection $\hat{P}_h$ admits the following estimate*

$$\|u - \hat{P}_h u\| \lesssim h^{r+1}\|u\|_{r+1},$$
$$\|u - \hat{P}_h u\|_1 \lesssim h^{r+1}\|u\|_{r+1}.$$

10.2.1 Two-grid discretizations

In this section, we shall present a number of algorithms for non-SPD problems based on two finite element spaces. The idea is to reduce a non-SPD problem into a SPD problem by solving a non-SPD problem on a much smaller space.

The basic mechanisms in our approach are two quasi-uniform triangulations of Ω, T_H and T_h, with two different mesh sizes H and h ($H > h$), and the corresponding finite element spaces $\mathcal{V}_H$ and $\mathcal{V}_h$ which will be called coarse and fine space respectively. In the applications given below, we shall always assume that

$$H = O(h^\lambda), \quad \text{for some } 0 < \lambda < 1. \tag{10.1}$$

With the bilinear form $\hat{A}$ defined in (10.3), for $h \ll 1$, let $u_h \in \mathcal{V}_h$ be the unique solution of

$$\hat{A}(u_h, \chi) = (f, \chi) \quad \forall\, \chi \in \mathcal{V}_h$$

and denote the bilinear form of the lower order terms of the operator $\hat{\mathcal{L}}$ (in (10.1)) by

$$N(v, \chi) = (\hat{A} - A)(v, \chi) = (\beta \cdot \nabla v, \chi) + (\gamma v, \chi).$$

Let us now present our first two-grid algorithm.

We note that the linear system in the second step of the above algorithm is SPD.

1. Find $u_H \in \mathcal{V}_H$ such that $\hat{A}(u_H, \varphi) = (f, \varphi) \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $A(u^h, \chi) + N(u_H, \chi) = (f, \chi) \quad \forall\, \chi \in \mathcal{V}_h$.

Theorem 64. *Assume $u^h \in \mathcal{V}_h$ is the solution obtained by Algorithm 10.2.1 for $H \ll 1$, then*

$$\|u_h - u^h\|_1 \lesssim H^{r+1}\|u\|_{r+1}$$

and

$$\|u - u^h\|_1 \lesssim (h^r + H^{r+1})\|u\|_{r+1}$$

provided that $u \in \mathcal{H}^{r+1}(\Omega)$.

Proof. A direct calculation and an application of Lemma 74 shows that

$$\begin{aligned} A(u_h - u^h, \chi) &= -N((I - \hat{P}_H)\, u_h, \chi) \\ &\lesssim \|(I - \hat{P}_H)u_h\|\, \|\chi\|_1 \\ &\lesssim (H\|u - u_h\|_1 + \|(I - \hat{P}_H)u\|)\, \|\chi\|_1 \\ &\lesssim\ H^{r+1}\, \|u\|_{r+1}\, \|\chi\|_1. \end{aligned}$$

The desired result then follows. □

Remark 3. If $\beta(x) = 0$ and $r \geq 2$, we have

$$\|P_h u - u^h\|_1 \lesssim \|u - u_H\|_{-1} \lesssim H^{r+2}\|u\|_{r+2}$$

and

$$\|u - u^h\|_1 \lesssim (h^r + H^{r+2})\|u\|_{r+2}.$$

Algorithm 10.2.1 can be applied in a successive fashion.

Let $u_h^0 = 0$; assume that $u_h^k \in \mathcal{V}_h$ has been obtained, $u_h^{k+1} \in \mathcal{V}_h$ is defined as follows
1. Find $e_H \in \mathcal{V}_H$ such that $\hat{A}(e_H + u_h^k, \varphi) = (f, \varphi) \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $A(u_h^{k+1}, \chi) + N(u_h^k + e_H, \chi) = (f, \chi) \quad \forall\, \chi \in \mathcal{V}_h$.

As it is well-known that most linear iterative methods for solving algebraic systems can be obtained by an appropriate matrix (or operator) splitting. For the nonsymmetric system under consideration, the most natural splitting would lead to the following iterative method

$$A(u_h^{k+1}, \chi) + N(u_h^k, \chi) = (f, \chi) \quad \forall\, \chi \in \mathcal{V}_h.$$

This iterative scheme, however, is not convergent in general. The Algorithm 10.2.1 may be considered as a modification of this "natural" iterative scheme with recourse to an additional coarse space.

Theorem 65. *Assume $u_h^k \in \mathcal{V}_h$ is the solution obtained by Algorithm 10.2.1 for $k \geq 1$, then*

$$\|u_h - u_h^k\|_1 \lesssim H^{k+r}\|u\|_{r+1},$$

and

$$\|u - u_h^k\|_1 \lesssim (h^r + H^{k+r})\|u\|_{r+1}$$

Proof. By definition and Lemma 74,

$$\begin{aligned} A(u_h - u_h^k, \chi) &= N((I - \hat{P}_H)(u_h^{k-1} - u_h), \chi) \\ &\le \|(I - \hat{P}_H)(u_h^{k-1} - u_h)\| \, \|\chi\|_1 \\ &\lesssim H \, \|u_h^{k-1} - u_h\|_1 \, \|\chi\|_1. \end{aligned}$$

This implies

$$\|u_h - u_h^k\|_1 \lesssim H\|u_h - u_h^{k-1}\|_1.$$

Applying the above estimate successively and then using Theorem 64 yield

$$\|u_h - u_h^k\|_1 \lesssim H^{k-1}\|u_h - u_h^1\|_1 \lesssim H^{k+r}\|u\|_{r+1}.$$

□

REMARK 4. The SPD system in the step of Algorithm 10.2.1 may not be solved exactly. The corresponding algorithms can be found in Xu [.xu-ncs,xu-ncs1.].

Before ending this section, we present an algorithm for symmetric and indefinite problem (namely $\beta(x) = 0$ in (10.1)). This algorithm is based on the following finite element space

$$\hat{\mathcal{V}}_h = (I - \hat{P}_H)\mathcal{V}_h.$$

1. Find $u_H \in \mathcal{V}_H$ such that $\hat{A}(u_H, \varphi) = (f, \varphi) \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $e_h \in \hat{\mathcal{V}}_h$ such that $A(e_h, \chi) = (f, \chi) \quad \forall\, \chi \in \hat{\mathcal{V}}_h$.
3. $u^h = u_H + e_h$.

We note that the system in the second step of Algorithm 10.2.1 is SPD. But since it is on the space $\hat{\mathcal{V}}_h$, this system may not be solved very easily. Nevertheless this algorithm is of certain theoretical interests. In fact, as shown in the next theorem,

$$\|u - u^h\|_1 \lesssim (h + H^3)\|u\|_2$$

if the linear finite element is used.

Theorem 66. *Assume $u^h \in \mathcal{V}_h$ is obtained by Algorithm 10.2.1, then*

$$\|u - u^h\|_1 \lesssim (h^r + H^{r+2})\|u\|_{r+1}.$$

Proof. As $\hat{A}$ is symmetric, so is $\hat{P}_H$. Thus

$$\hat{A}((I - \hat{P}_H)u, \chi) = (f, \chi) \quad \forall\, \chi \in \hat{\mathcal{V}}_h.$$

Therefore

$$A(u_h - (u_H + e_h), \chi) = -(\gamma(u - u_H), \chi) \lesssim H\|u - u_H\|\|\chi\|_1 \lesssim H^{r+2}\|u\|_{r+1}\|\chi\|_1.$$

where we have used the fact that $\|\chi\| \lesssim H\|\chi\|_1$ for $\chi \in \hat{\mathcal{V}}_h$. The desired result follows by taking $\chi = u_h - (u_H + e_h) \in \hat{\mathcal{V}}_h$. □

10.2.2 Iteration and precondition

Preliminaries

We assume that $\mathcal{V}$ is a given linear vector space $\mathcal{V}$ equipped with an inner product $(\cdot,\cdot)$. Let $\mathcal{V}$ denote the space of all linear operators from $\mathcal{V}$ to itself. We are interested in solving the equation

$$\hat{A}u = f, \tag{10.2}$$

for a given $f \in \mathcal{V}$. Here $\hat{A} \in \mathcal{V}$ is a given invertible operator satisfying

$$\hat{A} \;=\; A + N,$$

and $A \in \mathcal{V}$ is SPD in the sense that

$$(Au, v) = (u, Av) \quad \forall\, u, v \in \mathcal{V} \quad \text{and} \quad (Av, v) > 0 \quad \text{if} \quad v \neq 0;$$

the perturbation operator $N \in \mathcal{V}$ is not SPD in general.

As A is SPD, $(\cdot,\cdot)_A = (A\cdot,\cdot)$ defines an inner product on $\mathcal{V}$ and induces a norm on $\mathcal{V}$, denoted by $\|\cdot\|_A$. Given $G \in \mathcal{V}$, we define its A-norm by

$$\|G\|_A = \sup_{v\in\mathcal{V}} \frac{\|Gv\|_A}{\|v\|_A}.$$

The construction of an iterative algorithm for (10.2) often amounts to the construction of a $\hat{B} \in \mathcal{V}$ which behaves like $\hat{A}^{-1}$. One approach is to use $\hat{B}$ to obtain a linear iterative scheme as follows

$$u^{k+1} = u^k + \hat{B}(f - \hat{A}u^k), \tag{10.3}$$

for $k = 0, 1, 2, \cdots$, and any $u^0 \in \mathcal{V}$. Obvioulsly a sufficient condition for the convergence of scheme (10.3) is

$$\eta = \|I - \hat{B}\hat{A}\|_A < 1,$$

and in this case

$$\|u - u^k\|_A \le \eta^k \|u\|_A.$$

Another approach is to use $\hat{B}$ as a preconditioner for (10.2) in conjunction with GMRES type methods (c.f. [.gmres1,gmres2.]). Unlike the conjugate gradient method for SPD problem, the GMRES method may not be convergent without proper preconditioning. A preconditioner for the GMRES method is not only to speed up the convergence but more importantly to guarantee the convergence as well. More precisely, if there are two constants $\alpha_0, \alpha_1 > 0$ such that

$$(\hat{B}\hat{A}v, v)_A \ge \alpha_0 (v, v)_A, \quad \|\hat{B}\hat{A}v\|_A \le \alpha_1 \|v\|_A, \quad \forall\, v \in \mathcal{V},$$

then, the GMRES method applying to the preconditioned system

$$\hat{B}\hat{A}u = \hat{B}f$$

with the inner product $(\cdot,\cdot)_A$ converges at the rate $1 - \alpha_0^2/\alpha_1^2$ (cf. [.gmres1.]).

Now we assume that a subspace $\mathcal{V}_0 \subset \mathcal{V}$ is given, we define an operator $\hat{Z} : \mathcal{V}_0 \mapsto \mathcal{V}_0$, and three projections $Q_0, P_0, \hat{P}_z : \mathcal{V} \mapsto \mathcal{V}_0$ by, for all $u_0, v_0 \in \mathcal{V}_0$,

$$(\hat{Z}u_0, v_0) = (\hat{A}u_0, v_0),$$

and for all $u \in \mathcal{V}, v_0 \in \mathcal{V}_0$

$$(AP_0u, v_0) = (Au, v_0), \quad (\hat{A}\hat{P}_zu, v_0) = (\hat{A}u, v_0), \quad (Q_0u, v_0) = (u, v_0).$$

It is clear that $\hat{Z}$, P_0 and Q_0 are well defined. We shall assume that $\hat{Z}$ is invertible, which implies that $\hat{P}_z$ is also well-defined.

By the definitions of $\hat{P}_z, \hat{Z}$ and Q_0,

$$\hat{Z}\hat{P}_z = Q_0\hat{A}.$$

It follows that, for a given $f \in \mathcal{V}$,

$$\hat{u}_0 = \hat{Z}^{-1}Q_0f \quad \text{if and only if} \quad (\hat{A}\hat{u}_0, v_0) = (f, v_0), \quad \forall\ v_0 \in \mathcal{V}_0.$$

Many estimates in this paper will be established in terms of the following parameter

$$\delta_0 = \sup_{u,v\in\mathcal{V}} \frac{(N(I-\hat{P}_z)u, v)}{\|u\|_A\|v\|_A}. \tag{10.4}$$

The assumption that we shall make late is that δ_0 can be sufficiently small if the subspace $\mathcal{V}_0$ is properly chosen.

In the study of preconditioners, we need to use another parameter defined by

$$\bar{\delta} = \sup_{u,v\in\mathcal{V}} \frac{(Nu, v)}{\|u\|_A\|v\|_A}.$$

It is easy to see that

$$\|A^{-1}N\|_A \le \bar{\delta}. \tag{10.5}$$

Observe that $\bar{\delta} = \delta_0$ if $\mathcal{V}_0 = \{0\}$. Without loss of generality, we assume that $\delta_0 \le \bar{\delta}$.

Lemma 75. *For any $u \in \mathcal{V}$*

$$\|(\hat{P}_z - P_z)u\|_A \le \delta_0\|u\|_A, \quad \|u - \hat{P}_zu\|_A \le (1+\delta_0)\|u\|_A. \tag{10.6}$$

Proof. It follows from the definitions of $\hat{P}_z$ and P_z that

$$(A(\hat{P}_z - P_z)u, v_0) = (N(I-\hat{P}_z)u, v_0), \quad \forall\ u \in \mathcal{V}, v_0 \in \mathcal{V}_0,$$

which, with $v_0 = (\hat{P}_z - P_z)u$, implies the first inequality in (10.6). The second estimate obviously follows from the first one. □

Linear iterative algorithms

We now present the main algorithm proposed in Xu [.xu-ncs.]. The algorithm depends on a given solver, represented by a $B \in \mathcal{V}$, for A satisfying

$$\|I - BA\|_A < 1.$$

Like in the classic multigrid method, the first step of the above algorithm plays the role of correction on the small subspace $\mathcal{V}_0$; the second step plays the role of smoothing (by the SPD operator A).

Let us derive the error equation of the above algorithm. Without loss of generality, we assume that $p = 1$. Note that $f = \hat{A}u$, it follows that

$$\hat{u}_0 = \hat{P}_z(u - u^k) \quad \text{and} \quad v^1 = B\hat{A}(I-\hat{P}_z)(u-u^k).$$

Thus

$$u - u^{k+1} = (I - B\hat{A})(I - \hat{P}_z)(u - u^k).$$

Obviously the Algorithm 10.2.2 is identical to (10.3) if $\hat{B}$ satisfies

$$I - \hat{B}\hat{A} = (I - B\hat{A})(I - \hat{P}_z). \tag{10.7}$$

Given $u^0 \in \mathcal{V}$. Assume u^k is defined for $k \geq 0$, then

1. Solve (exactly) the equation on $\mathcal{V}_0$:
$$\hat{A}_0 \hat{u}_0 = Q_0(f - \hat{A}u^k).$$
2. Set $g = f - \hat{A}(u^k + \hat{u}_0)$, for $i = 0, 1, \cdots, p$ and $v^0 = 0$
$$v^{i+1} = v^i + B(g - Av^i),$$
3. $u^{k+1} = u^k + \hat{u}_0 + v^p$.

Theorem 67. *Assume that $\hat{B}$ is given by* (10.7)*, then*

$$\|I - \hat{B}\hat{A}\|_A \leq \eta,$$

where

$$\eta = \rho^p + 3\delta_0, \quad \rho = \|I - BA\|_A. \tag{10.8}$$

Consequently

$$\|u - u^k\|_A \leq (\rho^p + 3\delta_0)^k \|u - u^0\|_A,$$

where u^k are defined by Algorithm 10.2.2 and u is the solution of (10.2)*. Therefore the Algorithm 10.2.2 is convergent if δ_0 is sufficiently small so that $3\delta_0 < 1 - \rho^p$.*

Proof. Without loss of generality, we assume that $p = 1$. Given $u \in \mathcal{V}$, denote $u_0 = \hat{P}_z u$, $v = A^{-1}\hat{A}(u - u_0)$ and $w = u - u_0$. We shall first show that

$$\|w - v\|_A \leq \delta_0 \|u\|_A, \quad \|v\|_A \leq (1 + 2\delta_0)\|u\|_A. \tag{10.9}$$

In fact

$$\begin{aligned}\|w - v\|_A^2 &= (A(w - v), w - v) = ((A - \hat{A})(u - u_0), w - v) \\ &= -(N(u - u_0), w - v) \leq \delta_0 \|u\|_A \|w - v\|_A.\end{aligned}$$

The first estimate in (10.9) then follows. To see the second estimate in (10.9), by Lemma 2.3

$$\begin{aligned}\|v\|_A^2 &= (Av, v) = (\hat{A}(u - u_0), v) = (A(u - u_0), v) + (N(u - u_0), v) \\ &\leq (1 + \delta_0)\|u\|_A \|v\|_A + \delta_0 \|u\|_A \|v\|_A \leq (1 + 2\delta_0)\|u\|_A \|v\|_A.\end{aligned}$$

Therefore (10.9) is justified.

Thanks to (10.9), the rest of the proof is easy:

$$\begin{aligned}&\|(I - B\hat{A})(I - \hat{P}_z)u\|_A = \|w - B(Av)\|_A \\ &\leq \|w - v\|_A + \|v - B(Av)\|_A \leq \delta_0 \|u\|_A + \rho \|v\|_A \\ &\leq (\delta_0 + \rho(1 + 2\delta_0))\,\|u\|_A \leq (\rho + 3\delta_0)\,\|u\|_A\end{aligned}$$

as desired. □

Preconditioners for GMRES type methods

Based on the theory just developed, a number of preconditioners can be derived in a straightforward fashion. In particular the preconditioners presented in Xu and Cai [.xu cai.] can be obtained easily with weaker assumptions.

First, as a direct consequence of Theorem 2.1, we have

Theorem 68.

$$\hat{B} = (I - B\hat{A})\hat{A}_0^{-1}Q_0 + B, \tag{10.10}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \geq (1-\eta)(v,v)_A, \quad \|\hat{B}\hat{A}v\|_A \leq (1+\eta)\|v\|_A.$$

The proof of the above theorem is straightforward and hence omitted.

We shall now derive the theory developed in [.xu cai.].

Theorem 69. *Let*

$$\hat{B} = \omega\hat{A}_0^{-1}Q_0 + B. \tag{10.11}$$

Then, for η given by (10.8) *and for all $v \in \mathcal{V}$*

$$(\hat{B}\hat{A}v, v)_A \geq \frac{1}{2}(1-\eta)(v,v)_A, \quad \|\hat{B}\hat{A}v\|_A \leq (\omega+2)(1+\bar{\delta})\|v\|_A, \tag{10.12}$$

provided that ω is sufficiently large and δ_0 is sufficiently small, e.g.

$$\omega \geq \frac{(1+2\bar{\delta})^2}{1-\eta}, \quad \delta_0 \leq \frac{1}{4}\frac{1-\eta}{\omega+1+2\bar{\delta}}. \tag{10.13}$$

Proof. Obviously

$$\begin{aligned}\hat{B}\hat{A} &= \omega\hat{P}_0 + B\hat{A} = (\omega - 1 + B\hat{A})\hat{P}_0 + \hat{P}_0 + B\hat{A}(I-\hat{P}_0)\\ &= (\omega-1+B\hat{A})P_0 + (\omega-1+B\hat{A})(\hat{P}_0 - P_0) + \hat{P}_0 + B\hat{A}(I-\hat{P}_0)\end{aligned}$$

By (10.5) and the fact that $\|I - BA\|_A < 1$, it is easy to show that

$$\|I - B\hat{A}\|_A \leq 1 + 2\bar{\delta}.$$

Hence, by (10.6)

$$((\omega-1+B\hat{A})(\hat{P}_0 - P_0)v, v)_A \leq (\omega+1+2\bar{\delta})\delta_0\|v\|_A^2.$$

An application of Cauchy-Schwarz inequality gives

$$\begin{aligned}((I-B\hat{A})P_0v, v)_A &\leq \|I - B\hat{A}\|_A\|P_0v\|_A\|v\|_A\\ &\leq \frac{1}{1-\eta}(1+2\bar{\delta})^2\|P_0v\|_A^2 + \frac{1-\eta}{4}\|v\|_A^2.\end{aligned}$$

Combining the above two estimates with Theorem 68 yields

$$(\hat{B}\hat{A}v, v)_A \geq (\omega - \frac{(1+2\bar{\delta})^2}{1-\eta})\|P_0v\|_A^2 + \left(\frac{3(1-\eta)}{4} - (\omega+1+2\bar{\delta})\delta_0\right)\|v\|_A^2.$$

The first estimate in (10.12) then follows if (10.13) holds. The rest of the proof is straigtforward. □

We are now in a position to derive the main result in [.xu cai.].

Theorem 70. *Assume that $\bar{B}$ is a SPD preconditioner for A and*

$$\hat{B} = \omega\hat{A}_0^{-1}Q_0 + \bar{B}. \tag{10.14}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \geq \frac{\lambda_0 + \lambda_1}{4}(\frac{2\lambda_0}{\lambda_1 + \lambda_0} - 3\delta_0)A(v, v),$$

and

$$\|\hat{B}\hat{A}v\|_A \leq (\omega + 2)(1 + \bar{\delta})\frac{\lambda_0 + \lambda_1}{2}\|v\|_A,$$

provided that ω is sufficiently large and δ_0 is sufficiently small. Here

$$\lambda_0 = \lambda_{\min}(BA), \lambda_1 = \lambda_{\max}(BA).$$

Proof. Let $B = \frac{2}{\lambda_0+\lambda_1}\bar{B}$. Then

$$\rho = \|I - BA\|_A \leq \frac{\lambda_1 - \lambda_0}{\lambda_1 + \lambda_0} < 1.$$

The desired result can be derived from Theorem 69. □

Subspace correction method

The algorithm we have studied above are based on a given iterative algorithm for the SPD problem. In this section, we shall discuss a special class of iterative methods for SPD problem and discuss the corresponding Algorithm 10.2.2 and its modification.

Suppose that $\mathcal{V}_0$ used in the definition of Algorithm 10.2.2 coincides with that in the decomposition (**??**). Then, if the Algorithm 10.2.2 is applied with Algorithm **??**, the subspace problems on $\mathcal{V}_0$ are solved twice in each iteration, once for A_z and once for $\hat{Z}$. We shall remove the solver for A_0 from Algorithm **??** and modify the Algorithm 10.2.2 as follows:

Given $u^0 \in \mathcal{V}$. Assume that u^k is defined for $k \geq 1$, then we define $u^{k+1} = \hat{u}^k + v^J$ where

$$\hat{u}^k = u^k + \hat{Q}_z Z^{-1}(f - \hat{A}u^k)$$

and, for $i = 1, \cdots, J$,

$$v^i = v^{i-1} + R_i Q_i (g - Av^{i-1})$$

with $g = f - \hat{A}\hat{u}^k$ and $v^0 = 0$.

The error equation of the above algorithm is

$$u - u^{k+1} = (I - \hat{A})(I - \hat{P}_z)(u - u^k)$$

where

$$I - A = (I - T_J)(I - T_{J-1}) \cdots (I - T_1). \tag{10.15}$$

Theorem 71. *Assume that $\omega_1 < 2$. Then the Algorithm 10.2.2 converges if δ_0, given by* (10.4)*, is sufficiently small. Furthermore the error operator $\tilde{E} = (I - \hat{A})(I - \hat{P}_z)$ satisfies*

$$\|\tilde{E}\|_A \leq \eta$$

where

$$\eta = 1 + 5\delta_0 - \frac{2 - \omega_1}{K_0(1 + K_1)^2}. \tag{10.16}$$

Proof. Obviously, for B defined by (**??**) with $R_0 = A_0^{-1}$

$$I - BA = (I - A)(I - P_0). \tag{10.17}$$

A direct manipulation yields

$$\begin{aligned}(I - \hat{A})(I - \hat{P}_z) &= (I - B\hat{A})(I - \hat{P}_z)\\ &+ (I - A)(P_z - \hat{P}_z) + (B-)N(I - \hat{P}_z).\end{aligned}$$

Thus

$$\begin{aligned}&\|(I - \hat{A})(I - \hat{P}_z)u\|_A \le \|(I - B\hat{A})(I - \hat{P}_z)u\|_A\\ &+ \|(I - A)(P_z - \hat{P}_z)u\|_A + \|(B-)N(I - \hat{P}_z)u\|_A\\ &\equiv I_1 + I_2 + I_3.\end{aligned}$$

The estimate of I_1 is given by Theorem 67

$$I_1 \le (\rho + 3\delta_0)\|u\|_A$$

where $\rho = 1-\frac{2-\omega_1}{K_0(1+K_1)^2}$ by Theorem **??**. By the assumption on R_i, $\|I - T_i\|_A \le 1$ which implies that $\|I - A\|_A \le 1$. Hence, by (10.6)

$$I_2 \le \|(P_z - \hat{P}_z)u\|_A \le \delta_0\|u\|_A.$$

It remains to estimate I_3. We first note that, by (10.17), $(B-)A = (I - A)P_z$. Thus

$$\|(B-)A\|_A = \|I - A\|_A\|P_z\|_A \le 1.$$

Let "t" and "$*$" denote the transpositions with respect to the inner products $(\cdot,\cdot)$ and $(A\cdot,\cdot)$ respectively, then

$$\|(B-)^tA\|_A = \|[(B-)A]^*\|_A = \|(B-)A\|_A \le 1.$$

Consequently

$$\begin{aligned}&\|(B-)N(I - \hat{P}_z)u\|_A^2\\ &= ((B-)N(I - \hat{P}_z)u, A(B-)N(I - \hat{P}_z)u)\\ &\le \delta_0\|u\|_A\|(B-)^tA(B-)N(I - \hat{P}_z)u\|_A\\ &\le \delta_0\|u\|_A\|(B-)^tA\|_A\|(B-)N(I - \hat{P}_z)u\|_A\\ &\le \delta_0\|u\|_A\|(B-)N(I - \hat{P}_z)u\|_A.\end{aligned}$$

Hence

$$I_3 = \|(B-)N(I - \hat{P}_z)u\|_A \le \delta_0\|u\|_A.$$

The desired estimate then follows. □

With the subspace correction methods for the SPD problem, we shall now disucss the corresponding preconditioners studied in Section 3.2.

Theorem 72. *For $\tilde{B}$ given by* (10.15)*, we have*

$$\hat{B} = (I - \tilde{B}\hat{A})\hat{A}_0^{-1}Q_0 + \tilde{B}, \tag{10.18}$$

Then, for η given by (10.16) *and for all $v \in \mathcal{V}$*

$$(\hat{B}\hat{A}v, v)_A \ge (1 - \eta)(v, v)_A, \quad \|\hat{B}\hat{A}v\|_A \le (1 + \eta)\|v\|_A.$$

Because of Theorem 71, the proof of this theorem or the next one is identical to that of Theorem 68 or Theorem 69.

Theorem 73. *For* $\tilde{B}$ *given by* (10.15)*, define*

$$\hat{B} = \omega \hat{A}_0^{-1} Q_0 + \tilde{B}. \tag{10.19}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \geq \frac{1}{2}(1-\eta)A(v, v),$$

and

$$\|\hat{B}\hat{A}v\|_A \leq (\omega + 2)(1 + \bar{\delta})\|v\|_A,$$

provided that ω *is sufficiently large and* δ_0 *is sufficiently small.*

Note preconditioner (10.19) may also be applied in the SPD case.

Theorem 74.

$$\hat{B} = \omega \hat{A}_0^{-1} Q_0 + \sum_{i=1}^{J} R_i Q_i. \tag{10.20}$$

Then, for all $v \in \mathcal{V}$

$$(\hat{B}\hat{A}v, v)_A \geq \frac{\lambda_0 + \lambda_1}{4}\left(\frac{2\lambda_0}{\lambda_1 + \lambda_0} - 4\delta_0\right)(v, v)_A,$$

and

$$\|\hat{B}\hat{A}v\|_A \leq (\omega + 2)(1 + \bar{\delta})\frac{\lambda_0 + \lambda_1}{2}\|v\|_A,$$

provided that ω *is sufficiently large and* δ_0 *is sufficiently small.*

Proof. Using the obvious identity

$$\hat{B}\hat{A} = \hat{P}_0 - P_0 + (\omega - 1)\hat{P}_0 + \bar{B}\hat{A},$$

the desired result then follows by (10.6) and Theorem 70. □

10.3 Nonlinear Problems

10.3.1 Introduction

The main purpose of this paper is to present some discretization techniques based on two (or more) finite element subspaces for solving partial differential equations (PDE). Examples under our study here for this technique are linear as well as nonlinear second order elliptic boundary value problems. Inspired by Xu [.xu-ncs,xu-ncs1.] for a method to solve nonsymmetric and indefinite linear algebraic systems, we employ two finite element subspaces, $\mathcal{V}_H$ and $\mathcal{V}_h$ (with mesh size $h \ll H$), in our discretization schemes. On the coarser space $\mathcal{V}_H$, we use the standard finite element discretization to obtain a rough approximation $u_H \in \mathcal{V}_H$ and then solve a linearized equation based on u_H to produce a corrected solution $u^h \in \mathcal{V}_h$. A remarkable fact about this simple technique is that the space $\mathcal{V}_H$ can be extremely coarse (in contrast to $\mathcal{V}_h$) to still maintain the optimal accuracy. For example, if the piecewise linear finite element is used for a semilinear equation, u^h is asymptotically as accurate as the standard (nonlinear) finite element discretization in the finer space $\mathcal{V}_h$ if $H = O(h^{\frac{1}{4}})$. Moreover, if two linearized systems are solved on $\mathcal{V}_h$, it suffices to take $H = O(h^{\frac{1}{8}})$. This means that solving a nonlinear equation is not much more difficult than solving one linear equation, since $\dim \mathcal{V}_H \ll \dim \mathcal{V}_h$ and the work for solving u_H is relatively negligible.

The two-grid algorithm is also extended to multiple subspaces $\mathcal{V}_{h_j}$ and optimal estimates are obtained with, for example, $h_1 = O(H^4|\log H|)$ and $h_j = h_{j-1}^2|\log h_{j-1}|$ $(2 \le j \le J)$. This type of algorithm is related to the so-called *projective Newton's method* studied by Vainikko [.vainikko.] and Witsch [.witsch.]. A convergence analysis of this algorithm was given in Rannacher [.RC80.] and recently in [.R91.]. Similar methods have also been studied by Bank [.bank nonlinear.] for the multigrid iterative solution of the nonlinear algebraic systems resulted from the standard finite element discretization. For other multigrid methods for nonlinear problems, we refer, for example, to Brandt [.br1.], Hackbusch [.Ha85.] and Reusken [.Re88.] and the references cited therein.

Our method is also in a way related to the so-called *mesh independence principle* (MIP) that has been studied, for example, in [.AMP79, ABPR86, AB87, DP90, DET82, R91.] (and other references cited therein) for solving nonlinear differential equations by the Newton iterations. MIP refers to the fact that the number of Newton iteration in solving the discretized (by finite difference or finite element) nonlinear differential equation is asymptotically independent of the discretization parameters such as the mesh size. With the method in this paper, a stronger MIP holds: only one Newton iteration is sufficient. (In this statement, of course, we did not consider the number of Newton iterations needed to solve the nonlinear system from the coarse grid, which requires very little work as compared with the one Newton iteration in the fine grid.)

The error analysis of our two-grid methods is based on some $\mathcal{L}^p$ and $\mathcal{W}_p^1$ estimates $(2 \le p \le \infty)$ for the standard finite element discretization. The case $p = 2$ and $p = \infty$ have been studied by many authors (cf. Schultz [.schultz 1971.], Douglas and Dupont [.DD75.], Nitsche [.NN77.], Johnson and Thomée [.JT75.], Rannacher [.Rannacher 1977.], Frehse [.Frehse eine 1976.], Mittelmann [.mittelmann 1977.], Frehse and Rannacher [.FR76,FR78.], Rannacher [.Rannacher Calcolo.], Nitsche [.NN76.], Dobrowolski and Rannacher [.DR80.]), but the case $p \neq 2$ or ∞ can not be found in the literature. Because of their independent theoretical interests (in addition to the application in this paper), we shall give detailed derivation of these estimates (in §3). One main idea in our analysis is to linearize the nonlinear partial differential equations at the exact solution and consider its finite element discretizations. Such an idea has been used in most of the aforementioned papers, but our analysis appears to be much simpler. One observation that plays a significant role in our analysis is that the finite element approximation of the nonlinear equation is "super-close" to the finite element approximation to the aforemented linearized equation. As a result, the analysis of nonlinear problems is, in a straightforward fashion, reduced to the analysis of linear problems. For this reason, we shall give a brief presentation on the finite element theory for linear problems (in §2) and in particular on some estimates for some discrete Green's functions.

It is also observed that that the the finite element solution of a nonsymmetric and/or indefinite linear equation is "super-close" to the finite element solution of some symmetric positive definite equation. These facts are useful for both the theoretical analysis and the design of efficient solvers for the resultant algebraic systems. In fact, we shall present several algorithms based on such considerations for nonsymmetric and/or indefinite linear systems (in §4).

For simplicity of exposition, only the scalar equations will be considered in this paper, but the techniques and the corresponding results are extended to certain systems of equations in a very straightforward fashion. For clarity of presentation, we will only consider two dimensional problems as many results for three dimensional linear problems needed in our nonlinear analysis are not readily available in the literature. It is well-known that, in the finite element theory, special care needs to be taken near the curved part of the boundary in order to achieve the best approximation for higher order elements. For the sake of simplicity, we will not get into the technical details along this direction and our presentation will be made solely for polygonal domains. For technical reasons for some of the results in the paper, we will further assume the polygonal domain is also convex.

Therefore, we assume that Ω is a convex polygonal domain in the plane. For $p \ge 1$ and integer $m \ge 0$, let $\mathcal{W}_p^m(\Omega)$ be the standard Sobolev space with a norm $\|\cdot\|_{m,p}$ given by

$$\|v\|_{m,p}^p = \sum_{|\alpha|\le m} \|D^\alpha v\|_{\mathcal{L}^p(\Omega)}^p.$$

For $p = 2$, we denote $\mathcal{H}^m(\Omega) = \mathcal{W}_2^m(\Omega)$ and $\mathcal{H}_0^1(\Omega) = \{v \in \mathcal{H}^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of *trace*, $\|\cdot\|_m = \|\cdot\|_{m,2}$ and $\|\cdot\| = \|\cdot\|_{0,2}$. The space $\mathcal{H}^{-1}(\Omega)$, the dual of $\mathcal{H}_0^1(\Omega)$, will also be used.

Throughout this paper, we shall use the letter C or c (with or without subscripts) to denote a generic positive constant which may stand for different values at its different appearances. When it is not important to keep track of these constants, we shall conceal the letter C or c into the notation $\lesssim$ or $\gtrsim$. Here

$$x \lesssim y \quad \text{means} \quad x \le Cy \quad \text{and} \quad x \gtrsim y \quad \text{means} \quad x \ge cy.$$

10.3.2 $\mathcal{L}^p$ estimates for nonlinear problems

This section is devoted to the standard finite element discretization for nonlinear elliptic boundary value problems. Existence and uniqueness of the finite element solution will be discussed and error estimates in $\mathcal{L}^p$ nomrs will be derived.

A model problem and its finite element discretization

We consider the following second order quasi-linear elliptic problem:

$$\begin{cases} -\mathrm{div}(F(x,u,\nabla u)) + g(x,u,\nabla u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.1}$$

We assume that $F(x,y,z) : \overline{\Omega}\times\mathbb{R}^1\times\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ and $g(x,y,z) : \overline{\Omega}\times\mathbb{R}^1\times\mathbb{R}^2 \longrightarrow \mathbb{R}^1$ are smooth functions and that (10.1) has a solution $u \in \mathcal{H}^1_0(\Omega)\cap\mathcal{W}^2_{2+\epsilon}(\Omega)$ (for some $\varepsilon > 0$).

For any $w \in \mathcal{W}^1_\infty(\Omega)$, we denote

$$a(w) = D_zF(x,w,\nabla w) \in \mathbb{R}^{2\times 2}, \quad b(w) = D_yF(x,w,\nabla w) \in \mathbb{R}^2,$$
$$c(w) = D_zg(x,w,\nabla w) \in \mathbb{R}^2, \quad d(w) = D_yg(x,w,\nabla w) \in \mathbb{R}^1.$$

The linearized operator $\mathcal{L}$ at w (namely the Fréchet derivative of $\mathcal{L}$ at w) is then given by

$$\mathcal{L}'(w)v = -\mathrm{div}(a(w)\nabla v + b(w)v) + c(w)\nabla v + d(w)v.$$

Our basic assumptions are, first of all, for the solution u of (10.1)

$$\xi^T a(u)\xi \ge \alpha_0|\xi|^2 \quad \forall\, \xi \in \mathbb{R}^2, \quad x \in \overline{\Omega}$$

for some constant $\alpha_0 > 0$ and secondly $\mathcal{L}'(u) : \mathcal{H}^1_0(\Omega) \to \mathcal{H}^{-1}(\Omega)$ is an isomorphism. As a result of these assumptions, u must be an isolated solution of (10.1).

For convenience of exposition, we introduce two parameters δ_1 and δ_2 as follows.

$$\delta_2 = \begin{cases} 0 \text{ if } D_z^2F(x,y,z) \equiv 0, D_z^2g(x,y,z) \equiv 0 \\ 1 \text{ otherwise} \end{cases}$$

and

$$\delta_1 = \begin{cases} 0 \text{ if } \delta_2 = 0,\ D_yD_zF(x,y,z) \equiv 0,\ D_yD_zg(x,y,z) \equiv 0 \\ 1 \text{ otherwise.} \end{cases}$$

If $\delta_2 = 0$ and $\delta_1 = 1$, (10.1) is mildly nonlinear for which

$$\mathcal{L}(u) = -\mathrm{div}(\alpha(x,u)\nabla u + \beta(x,u)) + \gamma(x,u)\cdot\nabla u + g(x,u).$$

If $\delta_1 = \delta_2 = 0$, (10.1) is semilinear for which

$$\mathcal{L}(u) = -\mathrm{div}(\alpha(x)\nabla u + \beta(x,u)) + g(x,u). \tag{10.2}$$

Setting
$$A(v,\varphi) = (F(\cdot,v,\nabla v),\nabla\varphi) + (g(\cdot,v,\nabla v),\varphi),$$
then the solution u of (10.1) satisfies
$$A(u,\chi) = 0 \quad \forall\, \chi \in \mathcal{V}.$$

The classic finite element approximation of (10.1) is to find $u_h \in \mathcal{V}_h$ such that

$$A(u_h,\chi) = 0 \quad \forall\, \chi \in \mathcal{V}_h. \tag{10.3}$$

Introducing the bilinear form (induced by $\mathcal{L}'(w)$)

$$A'(w;v,\varphi) = (a(w)\nabla v + b(w)v,\nabla\varphi) + (c(w)\cdot\nabla v + d(w)v,\varphi),$$

we have

Lemma 76. *For any $v, v_h, \chi \in \mathcal{V}$,*

$$A(v_h,\chi) = A(v,\chi) + A'(v;v_h - v,\chi) + R(v,v_h,\chi). \tag{10.4}$$

Thus $u_h \in \mathcal{V}_h$ solves (10.3) *if and only if*

$$A'(u;u - u_h,\chi) = R(u,u_h,\chi) \quad \forall\, \chi \in \mathcal{V}_h, \tag{10.5}$$

where the remainder R, for given $K > 0$ and the functions v and v_h satisfying $\|v\|_{1,\infty} + \|v_h\|_{1,\infty} \le K$, satisfies the estimate

$$|R(v,v_h,\chi)| \le C(\|e_h\|_{0,2p}^2 + \delta_1\|e_h\nabla e_h\|_{0,p} + \delta_2\|\nabla e_h\|_{0,2p}^2)\,\|\nabla\chi\|_{0,q} \tag{10.6}$$

where the constant C depend on K and $e_h = v - v_h$, $1/p + 1/q = 1$, $p,q \ge 1$.

Proof. Set $\eta(t) = A(v + t(v_h - v),\chi)$. (10.4) follows from the elementary identity

$$\eta(1) = \eta(0) + \eta'(0) + \int_0^1 \eta''(t)(1-t)\,dt$$

with

$$R(v,v_h,\chi) = \int_0^1 \eta''(t)(1-t)\,dt.$$

A straightforward calculation shows that

$$\begin{aligned}\eta''(t) &= ((F_{zz}\nabla e_h)\nabla e_h + 2F_{yz}\nabla e_h e_h + F_{yy}e_h^2,\nabla\chi)\\ &\quad + (g_{zz}\nabla e_h\cdot\nabla e_h + 2g_{yz}\nabla e_h e_h + g_{yy}e_h^2,\chi).\end{aligned}$$

The estimate for R follows with an appropriate constant C satisfying

$$C \ge \max_{x\in\bar\Omega,|y|\le K,|z|\le K} (|F_{zz}| + 2|F_{yz}| + |F_{yy}| + |g_{zz}| + 2|g_{yz}| + |g_{yy}|).$$

□

Lemma 77. *Let $\hat P_h$ be the projection with respect to the bilinear form $A'(u;\cdot,\cdot)$. Then, for $h \ll 1$, the finite element equation* (10.3) *has a solution u_h satisfying*

$$\|u_h - \hat P_h u\|_{1,\infty} \le h^\sigma \ \ \& \ \ and \|u - u_h\|_{1,\infty} \lesssim h^\sigma \tag{10.7}$$

for some $\sigma > 0$. Furthermore there exists a constant $\eta > 0$ such that u_h is the only solution satisfying

$$\|u_h - u\|_{1,\infty} \le \eta.$$

Similar existence and uniqueness result can be found in the literature, cf. Rannacher [.rannacher calcolo.]. For completeness, we shall now include a simple proof.

Proof. Define a nonlinear operator $\Phi : \mathcal{V}_h \to \mathcal{V}_h$ by

$$A'(u; \Phi(v_h), \chi) = A'(u; u, \chi) - R(u, v_h, \chi) \quad \forall\, \chi \in \mathcal{V}_h.$$

By (10.7) it is easy to see that Φ is continuous. As $u \in \mathcal{H}_0^1(\Omega) \cap \mathcal{W}^2_{2+\varepsilon}(\Omega)$, there exists a $\sigma > 0$ (by Lemma 74) such that

$$\|u - \hat{P}_h u\|_{1,\infty} \lesssim h^{\sigma}.$$

Defining a set

$$B = \{v \in \mathcal{V}_h : \ \ \|v - \hat{P}_h u\|_{1,\infty} \le h^{\sigma}\},$$

we claim that $\Phi(B) \subset B$ for $h \ll 1$. In fact

$$A'(u; \Phi(v_h) - \hat{P}_h u, \chi) = -R(u, v_h, \chi).$$

For $v_h \in B$, taking $\chi = \hat{g}_h^z$ (defined by (**??**) with $\hat{A}(\cdot,\cdot) = A'(u; \cdot, \cdot)$) gives

$$\begin{aligned}
\|\Phi(v_h) - \hat{P}_h u\|_{1,\infty} &\le C_0 |\log h|\ \|u - v_h\|_{1,\infty}^2 \\
&\le 2C_0 |\log h| \left(\|u - \hat{P}_h u\|_{1,\infty}^2 + \|v_h - \hat{P}_h u\|_{1,\infty}^2\right) \\
&\le C_1 |\log h|\ h^{2\sigma} \le h^{\sigma}
\end{aligned}$$

if $h \ll 1$. Thus $\Phi(B) \subset B$. An application of Brouwer's fixed point theorem shows the existence of a $u_h \in B$ so that $\Phi(u_h) = u_h$. By definition such a u_h satisfies the desired properties.

To prove the uniqueness, let u_h and $\tilde{u}_h$ be two solutions of (10.3) satisfying

$$\|u - u_h\|_{1,\infty} \le \eta \ \text{ and } \ \|u - \tilde{u}_h\|_{1,\infty} \le \eta.$$

Then

$$\int_0^1 A'(u_h + t(\tilde{u}_h - u_h); \tilde{u}_h - u_h, \chi)dt = A(\tilde{u}_h, \chi) - A(u_h, \chi) = 0.$$

By assumption and Lemma 71, if $\eta \ll 1$

$$\|u_h - \tilde{u}_h\|_1 \lesssim \sup_{\chi \in \mathcal{V}_h} \frac{\int_0^1 A'(u_h + t(\tilde{u}_h - u_h); \tilde{u}_h - u_h, \chi)dt}{\|\chi\|_1} = 0.$$

Thus $\tilde{u}_h = u_h$. □

$\mathcal{L}^p$ estimates

We shall now derive some $\mathcal{L}^p$ estimates for the finite element approximation. The main ingredient in our approach is the following superconvergence estimate between nonlinear and linearized problems.

Lemma 78. *Assume that u_h and $\hat{P}_h$ are as described in Lemma 77. Then*

$$\|u_h - \hat{P}_h u\|_1 \lesssim \|u - u_h\|_{0,4}^2 + \delta_1 \|\nabla(u - u_h)^2\| + \delta_2 \|u - u_h\|_{1,4}^2, \tag{10.8}$$

$$\begin{aligned}
\|u_h - \hat{P}_h u\|_{1,\infty} \lesssim |\log h| \Big(&\|u - u_h\|_{0,\infty}^2 + \delta_1 \|\nabla(u - u_h)^2\|_{0,\infty} \\
&+ \delta_2 \|u - u_h\|_{1,\infty}^2 \ \Big).
\end{aligned} \tag{10.9}$$

Proof. The estimate (10.8) is obtained by taking $\chi = \hat{P}_h u - u_h$ in (10.5) and applying (10.6). The second estimate is obtained by taking $\chi = \hat{g}_h^z$ in (10.5) with $\hat{A}(\cdot,\cdot) = A'(u;\cdot,\cdot)$. □

Theorem 75. *Assume that $u \in \mathcal{W}^2_{2+\epsilon}(\Omega)$ and $u_h \in \mathcal{V}_h$ are the solutions of (10.1) and (10.3), respectively, that satisfy (10.7). Then*

$$\|u - u_h\|_{1,p} \lesssim h^r \quad \text{if } u \in \mathcal{W}^{r+1}_p(\Omega), \quad 2 \le p \le \infty, \tag{10.10}$$

$$\|u - u_h\|_{0,p} \lesssim h^{r+1} \quad \text{if } u \in \mathcal{W}^{r+1}_p(\Omega), \quad 2 \le p < \infty, \tag{10.11}$$

and

$$\|u - u_h\|_{0,\infty} \lesssim h^{r+1}|\log h| \quad \text{if } u \in \mathcal{W}^{r+1}_\infty(\Omega). \tag{10.12}$$

Proof. We shall divide the proof for (10.10) into five different cases: $p = 2$, $p = \infty$, $p = 4$, $p \ge 4$ and $2 < p < 4$. It follows from (10.8) and (10.7) that

$$\|\hat{P}_h u - u_h\|_1 \lesssim \|u - u_h\|^2_{1,4} \lesssim \|u - u_h\|_{1,\infty}\, \|u - u_h\|_1 \lesssim h^\sigma \|u - u_h\|_1.$$

Thus, if $h \ll 1$

$$\|u - u_h\|_1 \lesssim \|u - \hat{P}_h u\|_1 \lesssim \inf_{\chi \in \mathcal{V}_h} \|u - \chi\|_1.$$

This implies (10.10) for $p = 2$.

By (10.9)

$$\|u - u_h\|_{1,\infty} \lesssim \|u - \hat{P}_h u\|_{1,\infty} + |\log h| h^\sigma \|u - u_h\|_{1,\infty}.$$

Thus if $h \ll 1$,

$$\|u - u_h\|_{1,\infty} \lesssim \|u - \hat{P}_h u\|_{1,\infty} \lesssim \inf_{\chi \in \mathcal{V}_h} \|v - \chi\|_{1,\infty},$$

which implies that (10.10) holds for $p = \infty$ and, by (10.4), that

$$\|u - u_h\|_{1,\infty} \lesssim h^{r-2/p} \text{ if } u \in \mathcal{W}^{r+1}_p(\Omega) \text{ and } 2 \le p < \infty. \tag{10.13}$$

By (10.8) and (10.9), we have

$$\begin{aligned} \|u_h - \hat{P}_h u\|_{1,4} &\lesssim \|u_h - \hat{P}_h u\|^{1/2}_{1,2} \|u_h - \hat{P}_h u\|^{1/2}_{1,\infty} \\ &\lesssim \|u - u_h\|_{1,4} \|u - u_h\|_{1,\infty}. \end{aligned}$$

Thus, if $h \ll 1$, we have

$$\|u - u_h\|_{1,4} \lesssim \|u - \hat{P}_h u\|_{1,4} \lesssim \inf_{\chi \in \mathcal{V}_h} \|u - \chi\|_{1,4},$$

which implies (10.10) for $p = 4$.

Now, we assume that $4 \le p < \infty$. Again, by (10.8) and (10.9), we have, if $u \in \mathcal{W}^{r+1}_p(\Omega)$

$$\begin{aligned} \|u_h - \hat{P}_h u\|_{1,p} &\lesssim \|u_h - \hat{P}_h u\|^{2/p}_{1,2} \|u_h - \hat{P}_h u\|^{1-2/p}_{1,\infty} \\ &\lesssim \|u - u_h\|^{4/p}_{1,4} \|u - u_h\|^{2-4/p}_{1,\infty} \\ &\lesssim h^{4r/p} h^{(r-2/p)(2-4/p)} \lesssim h^{2r-1} \lesssim h^r. \end{aligned}$$

This proves (10.10) for $p \geq 4$.

Now, we assume that $2 < p < 4$. It follows from the previous inequalities, (10.4) and (10.13) that

$$\begin{aligned}\|u_h - \hat{P}_h u\|_{1,p} &\lesssim \|u - u_h\|_{1,4}^{4/p} \|u - u_h\|_{1,\infty}^{2-4/p} \\ &\lesssim \inf_{\chi \in \mathcal{V}_h} \|u - \chi\|_{1,4}^{4/p} \|u - u_h\|_{1,\infty}^{2-4/p} \\ &\lesssim h^{(r+1/2-2/p)4/p} h^{(r-2/p)(2-4/p)} \lesssim h^{2r-2/p} \lesssim h^r.\end{aligned}$$

This proves (10.10) for $2 < p < 4$. The proof for (10.10) is then complete.

The proof of (10.12) is easy, since, by (10.9), we have

$$\begin{aligned}\|u - u_h\|_{0,\infty} &\leq \|u - \hat{P}_h u\|_{0,\infty} + \|u_h - \hat{P}_h u\|_{1,\infty} \\ &\leq \|u - \hat{P}_h u\|_{0,\infty} + |\log h| \|u - u_h\|_{1,\infty}^2.\end{aligned}$$

To prove (10.11), we shall apply a duality augument. Consider the auxiliary problem: find $w \in \mathcal{H}_0^1(\Omega)$ such that

$$A'(u; v, w) = (\varphi, v) \quad \forall\, v \in \mathcal{H}_0^1(\Omega).$$

Given $2 \leq p < \infty$, set $q = p/(p-1) \in (1, 2]$. By Lemma 70

$$\|w\|_{2,q} \lesssim \|\varphi\|_{0,q} \quad \text{for } \varphi \in \mathcal{L}^q.$$

It follows that

$$\|w - P_h w\|_{1,q} \lesssim h \|w\|_{2,q} \lesssim h \|\varphi\|_{0,q},$$

If $s > 2p/(p+2)$, by (**??**) and a well-known Sobolev imbedding theorem, we have

$$\|P_h w\|_{1,s/(s-1)} \lesssim \|w\|_{1,s/(s-1)} \lesssim \|w\|_{2,q} \lesssim \|\varphi\|_{0,q}.$$

Consequently, with $s > 2p/(p+2)$, by Lemma 76

$$\begin{aligned}(u - u_h, \varphi) = A'(u; u - u_h, w) &= A'(u; u - u_h, w - P_h w) + A'(u; u - u_h, P_h w) \\ &\lesssim \|u - u_h\|_{1,p} \|w - P_h w\|_{1,q} + \|u - u_h\|_{1,2s}^2 \|P_h w\|_{1,s/(s-1)} \\ &\lesssim (h \|u - u_h\|_{1,p} + \|u - u_h\|_{1,2s}^2) \|\varphi\|_{0,q}\end{aligned}$$

Thus

$$\|u - u_h\|_{0,p} \lesssim h \|u - u_h\|_{1,p} + \|u - u_h\|_{1,2s}^2.$$

For $p = 2$, if $u \in \mathcal{W}_2^{r+1}(\Omega)$

$$\|u - u_h\|_{0,2} \lesssim h \|u - u_h\|_{1,2} + \|u - u_h\|_{1,2+\epsilon}^2 \lesssim h^{r+1}.$$

For $2 < p < \infty$, since $4p/(p+2) < p$, we have

$$\|u - u_h\|_{0,p} \lesssim h \|u - u_h\|_{1,p} + \|u - u_h\|_{1,p}^2.$$

This yields the estimate (10.11). □

10.3.3 Two-grid methods

This section is devoted to some discretization techniques based on two (or more than two) finite element subspaces. The first subsection is on some simple techniques for some mildly nonlinear equations and the rest of the section is devoted to some two-grid methods based on the Newton's method for general quasilinear equations.

Some simple two-grid methods

The techniques presented here are similar to those for algorithms for non-SPD linear problems in §4 and they will be applied to the following mildly quasilinear equation

$$(10.14)\qquad \begin{cases} -\mathrm{div}(\alpha(x,u)\nabla u) + \beta(x,u)) + \gamma(x,u)\cdot\nabla u + g(x,u) = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This equation is a special case of (10.1) (with $\delta_2 = 0$) with $F(x,y,z) = \alpha(x,y)z + \beta(x,y)$ and $g(x,y,z) = \gamma(x,y)\cdot z + g(x,y)$. We assume the early assumptions on (10.1) all hold here.

To state the algorithm, we define, for $u, v, \chi \in \mathcal{W}^1_\infty(\Omega) \cap \mathcal{H}^\infty_r(\otimes)$

$$\tilde{A}(u; v, \chi) = (\alpha(\cdot,u)\nabla v + \beta(\cdot,u), \nabla\chi) + (\gamma(\cdot,u)\cdot\nabla v + f(\cdot,u), \chi).$$

Our first algorithm is a nonlinear extension of Algorithm 10.2.1.

1. Find $u_H \in \mathcal{V}_H$ such that $A(u_H, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $\tilde{A}(u_H; u^h, \chi) = 0 \quad \forall\, \chi \in \mathcal{V}_h$.

Theorem 76. *Assume $u^h \in \mathcal{V}_h$ is obtained by Algorithm 10.3.3, then*

$$\begin{aligned} \|u_h - u^h\|_1 &\lesssim H^{r+1} && \text{if } u \in \mathcal{H}^{\nabla+\infty}(\otimes). \\ \|u_h - u^h\|_{1,\infty} &\lesssim H^{r+1}|\log h| && \text{if } u \in \mathcal{W}^{r+1}_\infty(\Omega). \end{aligned}$$

Proof. By definition, it follows that

$$\tilde{A}(u_H; u_h - u^h, \chi) = \tilde{A}(u_H; u_h, \chi) - \tilde{A}(u_h; u_h, \chi) \lesssim \|u_h - u_H\|_{-1}\ \|\chi\|_1.$$

Thus

$$\|u_h - u^h\|_1 \lesssim \|u_h - u_H\|_{-1} \le \|u_h - u_H\|.$$

The first estimate then follows by combining Lemma 72. Now let $\hat{g}^z_h$ be the Green's function defined as in (**??**) with $\hat{A}(\cdot,\cdot) = \tilde{A}(u;\cdot,\cdot)$. Then, with $e^h_H = u_h - u^h$,

$$\begin{aligned} \partial\, e^h_H(z) &= \tilde{A}(u; e^h_H, \hat{g}^z_h) = \tilde{A}(u; e^h_H, \hat{g}^z_h) - \tilde{A}(u_H; e^h_H, \hat{g}^z_h) + A(u_H, e^h_H, \hat{g}^z_h) \\ &\lesssim \|u_h - u_H\|_{0,\infty}\, \|e^h_H\|_{1,\infty}\, \|\hat{g}^z_h\|_{1,1} + \|u_h - u_H\|_{0,\infty} \|g^z_h\|_{1,1} \\ &\lesssim\ H^{r+1}|\log h|\, \|e^h_H\|_{1,\infty} + H^{r+1}|\log\ h|. \end{aligned}$$

This implies, for some constant $c > 0$,

$$\left(1 - c\, H^{r+1}|\log\ h|\right)\ \|e^h_H\|_{1,\infty} \lesssim H^{r+1}|\log\ h|.$$

The desired result then follows if H is so small that $H^{r+1}|\log\ h| \ll 1$ (see (10.1)). □

Remark 4. If $r \ge 2$, we could use the negative norm estimate to conclude that

$$\|u_h - u^h\|_1 \lesssim \|u_h - u_H\|_{-1} \lesssim H^{r+2}.$$

Next we shall present an algorithm resulted by combining Algorithm 10.14 with Algorithm 10.1. This algorithm reduces a nonlinear problem to a SPD linear problem and a nonlinear system of smaller size.

Define

$$A_s(u; v, \chi) = (\alpha(\cdot,u)\nabla v, \nabla\chi),$$

and

$$N(u; v, \chi) = (\beta(\cdot,u), \nabla\chi) + (\gamma(\cdot,u)\cdot\nabla v + f(\cdot,u), \chi).$$

1. Find $u_H \in \mathcal{V}_H$ such that $A(u_H, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $A_s(u_H; u^h, \chi) + N(u_H; u_H, \chi) = 0 \quad \forall\, \chi \in \mathcal{V}_h$.

Theorem 77. *Assume that $u^h \in \mathcal{V}_h$ is obtained by Algorithm 10.3.3, then*

$$\begin{aligned} \|u_h - u^h\|_1 &\lesssim H^{r+1} && \text{if } u \in \mathcal{H}^{\nabla+\infty}(\otimes), \\ \|u_h - u^h\|_{1,\infty} &\lesssim H^{r+1}|\log\, h| && \text{if } u \in \mathcal{W}^{r+1}_\infty(\Omega). \end{aligned}$$

Proof. By definition

$$\begin{aligned} A_s(u_H; u_h - u^h, \chi) &= A_s(u_H; u_h, \chi) - A_s(u_h; u_h, \chi) \\ &\quad -N(u_h; u_h, \chi) + N(u_H; u_H, \chi) \\ &\lesssim \|u_H - u_h\|\, \|\chi\|_1. \end{aligned}$$

The desired result then follows easily. □

Correction by one Newton's iteration on the fine space

Unlike in the last subsection, the techniques here apply to the general quasi-linear equation (10.1).
Our first algorithm, roughly speaking, is to use the coarse grid approximation as an initial guess for one Newton iteration on the fine grid.

1. Find $u_H \in \mathcal{V}_H$ such that $A(u_H, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $A'(u_H; u^h, \chi) = A'(u_H; u_H, \chi) - A(u_H, \chi) \quad \forall\, \chi \in \mathcal{V}_h$.

Lemma 79. *Assume that u^h is the solution obtained by Algorithm 10.3.3, then*

$$\begin{aligned} \|u_h - u^h\|_1 &\lesssim \|u_h - u_H\|^2_{0,4} + \delta_1\|(u_h - u_H)^2\|_{1,4} + \delta_2\|u_h - u_H\|^2_1, \\ \|u_h - u^h\|_{1,\infty} &\lesssim |\log h|(\|u_h - u_H\|^2_{0,\infty} + \delta_1\|(u_h - u_H)^2\|_{1,\infty} + \delta_2\|u_h - u_H\|^2_{1,\infty}). \end{aligned}$$

Proof. By definition and (10.4),

$$A'(u_H; u_h - u^h, \chi) = A'(u_H; u_h - u_H, \chi) + A(u_H, \chi) = -R(u_H, u_h, \chi).$$

The first estimate follows from Lemma 71 (with $\epsilon = H$), (10.7) and (10.6). The proof of the second estimate is similar to the proof of Theorem 76 by using (10.6). □
As a direct consequence of Lemma 79 and Theorem 10.1, we have

Theorem 78. *Assume u_h is the solution obtained by Algorithm 10.3.3. If $u \in \mathcal{W}^{r+1}_4(\Omega)$ or if, for $d = 3, \delta_2 = 1$ and $r = 1$, $u \in W^2_p(\Omega)$ for some $p > 6$, then*

$$\|u_h - u^h\|_1 \lesssim (H^{2r+2} + \delta_1 H^{2r+1} + \delta_2 H^{2r}) \lesssim H^{2r}$$

If $u \in \mathcal{W}^{r+1}_\infty(\Omega)$, then

$$\|u_h - u^h\|_{1,\infty} \lesssim (H^{2r+2} + \delta_1 H^{2r+1} + \delta_2 H^{2r})|\log h| \lesssim H^{2r}|\log h|$$

Thus, if $h = O(H^{2+2/r} + \delta_1 H^{2+1/r} + \delta_2 H^{2r})$

$$\|u - u^h\|_1 \lesssim h^r \text{ and } \|u - u^h\|_{1,\infty} \lesssim h^r|\log h|.$$

To see the efficiency of Algorithm 10.3.3, we single out a special case of the above theorem.

Corollary 7. *If the Algorithm 10.3.3 is applied to the semi-linear equation (10.2) with the linear finite element discretization, then*

$$\begin{aligned}\|u_h - u^h\|_1 &\lesssim H^4 \quad && \text{if } u \in \mathcal{W}^2_4(\Omega),\\ \|u_h - u^h\|_{1,\infty} &\lesssim H^4|\log h| && \text{if } u \in \mathcal{W}^2_\infty(\Omega).\end{aligned}$$

According to Corollary 7, in order to obtain the optimal (or nearly optimal) approximation for the discretization u^h, it suffices to take $H = O(h^{\frac{1}{4}})$. To get an idea numerically if the fine mesh size is $h = 2^{-16}$ which gives $\dim \mathcal{V}_h \approx 3.3 \times 10^9$, the coarse mesh size H could be $H = h^{1/4} = 1/16$ which gives $\dim \mathcal{V}_H \approx 225$.

Correction by two Newton's iterations on the fine space

The algorithms presented above can be greatly improved if one further Newton's iteration is carried out on $\mathcal{V}_h$.

Corresponding to Algorithm 10.3.3, we have

1. Find $u_H \in \mathcal{V}_H$ such that $A(u_H, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $\tilde{A}(u_H; u^h, \chi) = 0 \quad \forall\, \chi \in \mathcal{V}_h$.
3. Find $u^*_h \in \mathcal{V}_h$ such that $A'(u^h; u^*_h, \chi) = A'(u^h; u^h, \chi) - A(u^h, \chi) \quad \forall\, \chi \in \mathcal{V}_h$.

Corresponding to Algorithm 10.3.3, we have

1. Find $u_H \in \mathcal{V}_H$ such that $A(u_H, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_H$.
2. Find $u^h \in \mathcal{V}_h$ such that $A'(u_H; u^h, \chi) = A'(u_H; u_H, \chi) - A(u_H, \chi) \quad \forall\, \chi \in \mathcal{V}_h$.
3. Find $u - u_h s \in \mathcal{V}_h$ such that $A'(u^h; u^*_h, \chi) = A'(u^h; u^h, \chi) - A(u^h, \chi) \quad \forall\, \chi \in \mathcal{V}_h$.

With arguments similar to those in the preceding subsection, we can obtain various results as follows.

Lemma 80. *For both Algorithms 10.3.3 and 10.3.3,*

$$\|u_h - u^*_h\|_{1,\infty} \lesssim \|u_h - u^h\|^2_{1,\infty}.$$

Thus, for Algorithm 10.3.3

$$\|u_h - u^*_h\|_{1,\infty} \lesssim H^{2r+2}|\log h|^2,$$

and for Algorithm 10.3.3

$$\|u_h - u^*_h\|_{1,\infty} \lesssim (H^{4r+4} + \delta_1 H^{4r+2})|\log h|^2.$$

Theorem 79. *For Algorithm 10.3.3, if $h = O(H^{2+2/r})$, then*

$$\|u - u^*_h\|_{1,\infty} \lesssim h^r|\log h|.$$

If $h = O(H^2)$, then

$$\|u - u^*_h\|_{0,\infty} \lesssim h^{r+1}|\log h|^2.$$

Theorem 80. *For Algorithm 10.3.3 if $h = O(H^{4+4/r} + \delta_1 H^{4+2/r})$, then*

$$\|u - u_h^*\|_{1,\infty} \lesssim h^r |\log h|.$$

If $h = O(H^4 + \delta_1 H^{4-2/(r+1)})$, then

$$\|u - u_h^*\|_{0,\infty} \lesssim h^{r+1} |\log h|^2.$$

Again, to get an idea of the efficiency of Algorithm 10.3.3, we have

Corollary 8. *If the Algorithm 10.3.3 is applied to the semi-linear equation ((10.4)) with the linear finite element discretization, then*

$$\|u_h - u^h\|_{1,\infty} \lesssim H^8 |\log h|^2$$

provided that $u \in \mathcal{W}^2_\infty(\Omega)$.

Multilevel linearization

From the above discussions, it appears that the algorithm using two subspaces would suffice for most practical applications. Nevertheless the algorithm can be made more general and perhaps more robust if multiple subspaces are used.

Assume that we are given a sequence of subspaces

$$\mathcal{V}_i = \mathcal{V}_{h_i} \subset \mathcal{V} \quad 0 \le i \le J.$$

Conceivably, we have $h_J \ll h_{J-1} \ll \cdots \ll h_0 = H \ll 1$.

---

1. Find $u_0 \in \mathcal{V}_0$ such that $A(u_0, \varphi) = 0 \quad \forall\, \varphi \in \mathcal{V}_0$.
2. For $j = 1, 2, \cdots, J$, find $u_j \in \mathcal{V}_j$ such that

$$A'(u_{j-1}; u_j, \chi) = A'(u_{j-1}; u_{j-1}, \chi) - A(u_{j-1}, \chi) \quad \forall\, \chi \in \mathcal{V}_j.$$

The above algorithm is similar to the so-called projective Newton's method [.witsch, vainikko.].

Lemma 81. *If $H \ll 1$,*

$$\|u - u_j\|_{1,\infty} \lesssim h_j^r + |\log h_j|\, \|u - u_{j-1}\|^2_{1,\infty}.$$

Proof. Similar to the proof of Lemma 79, we have

$$A'(u_{j-1}; u - u_j, \chi) = -R(u_{j-1}, u, \chi) \quad \forall\, \chi \in \mathcal{V}_j.$$

The desired estimate can then be obtained in a way similar to proofs of Theorem 76 and Lemma 79. □

Theorem 81. *If $H \ll 1$, $h_1 = (H^{2+2/r} + \delta_1 H^{2+1/r})|\log H|^{1/r}$ and*

$$c|\log h_{j-1}|^{\frac{1}{r}} h_{j-1}^2 \le h_j < h_{j-1} \quad 2 \le j \le J$$

for some appropriate positive constant c. Then

$$\|u - u_j\|_{1,\infty} \le c_1 h_j^r, \quad 1 \le j \le J. \tag{10.15}$$

Proof. By Lemma 80, there exists a constant $c_0 > 0$ such that

$$\|u - u_j\|_{1,\infty} \le c_0(h_j^r + |\log h_{j-1}|\|u - u_{j-1}\|_{1,\infty}^2).$$

By Theorems 75 and 78, if $H \ll 1$, the estimate (10.15) holds for $j = 0, 1$ with some constant $c_1 > 0$. Without loss of generality, we may assume $c_1 = 2c_0$. Assume now that (10.15) is valid for $j - 1$, then

$$\|u - u_j\|_{1,\infty} \le c_0(h_j^r + c_1^2|\log h_{j-1}|h_{j-1}^{2r}) \le c_0(1 + c_1^2 c^{-1})h_j^r$$

By induction, (10.15) holds with $c = c_1^2 = 4c_0^2$. □

A weaker form of the estimate in the above theorem has been conjectured by Rannacher [.Rannacher Calcolo.] and Bank [.bank nonlinear.] and recently proved by Rannacher [.R91.].

Note that if Theorem 79 is applied to semilinear equations with linear finite element discretization, one may take $h_1 = H^4|\log H|$ and $h_2 = H^8|\log H|^2$.

Corollary 9. *If $H \ll 1$ and, for some some constants $\eta \in (0, 1)$*

$$\eta h_{j-1} \le h_j < h_{j-1}, \qquad 1 \le j \le J.$$

Then

$$\|u - u_j\|_{1,\infty} \lesssim h_j^r, \quad 0 \le j \le J.$$

A result similar to the above corollary was contained in Bank [.bank-nonlinear1.] on his multigrid method for solving the nonlinear Galerkin equation (10.3).

10.3.4 Concluding remarks

The algorithms studied in this paper are potentially efficient for solving a large class of linear and nonlinear problems. Although our presentation has been confined to the second order elliptic boundary value problems, the techniques can naturally be applied to other types of problems as well. Roughly speaking, different aspects of a complex problem can be treated by spaces of different scales. In the examples studied in this paper, a very coarse grid space is sufficient for some nonsymmetric problems that is dominated (in certain analytic sense) by its symmetric part and is also sufficient to handle the nonlinearity for some mildly nonlinear problems. Symmetry versus nonsymmetry and linearity versus nonlinearity may not make a substantial difference on the analytic level, but their numerical approximation may differ considerably. The two-grid methods studied in this paper provide a new approach to take the advantage of some "nice properties" hidden in a complex problem.

An important aspect of our two-grid algorithms is that they can be naturally applied together with multigrid and domain decomposition methods. Most domain decomposition methods, for example, are in certain sense two-grid methods. The set of subdomains gives rise to a natural coarse grid. Hence the domain decomposition techniques fit perfectly well with our algorithms and the coarse grid plays two different important roles in such an application. Similar arguments also apply to multigrid methods. Suppose we have a multiple subspaces $\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_j \subset \mathcal{H}_0^1$. Naturally we can choose $\mathcal{V}_H = \mathcal{V}_0$ (and $\mathcal{V}_h = \mathcal{V}_j$, of course) in our two-grid algorithms.

Applications of multigrid and domain decomposition methods with our two-grid methods for nonlinear problems are satisfying from both theoretical and practical point of views, since the systems on the fine grid are all linear and hence theories and numerical codes for linear problems can be adopted with few modifications.

The linear systems on the fine space in the algorithms presented in §4 are SPD and their solution methods have been well developed, we refer to Xu [.xuunify.] for a summary of these methods. The linear systems from the fine space on the algorithms in §5 are mostly not SPD and may be solved by combining the algorithms in §4. As a result, a nonlinear system on the fine space may be reduced to few SPD linear systems on the fine space together with some linear and nonlinear systems on the coarse space.

Some two-grid methods have been further improved for semilinear problems in Xu [.xu-tg1.]. Some numerical examples on the performances of these algorithms can be found in Xu [.xu-tg1.] and Lee and Xu [.lee-xu-1.].