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Linear elasticity and finite elements

13.1 Linear elasticity equations in three dimensions

We consider a homogeneous isotropic elastic body occupying a bounded domain $\Omega \in \mathbb{R}^3$ with boundary Γ decomposed into two parts Γ_D and Γ_N such that Γ_D has a positive area. Let the elastic body be acted upon by a volume load $f = (f_1, f_2, f_3)$ and boundary load $g = (g_1, g_2, g_3)$ on Γ_N. Furthermore, we assume that the body is fixed along Γ_D.

Let $u = (u_1, u_2, u_3)$ be the displacement of the elastic body. We assume that u is small. The linear elasticity theory shows that the displacement u should satisfy the following equilibrium equation:

$$\sigma = \lambda \mathrm{div} u I + 2\mu\epsilon(u) \quad \text{in } \Omega \tag{13.1}$$

$$-\mathrm{div}\sigma = f \quad \text{in } \Omega \tag{13.2}$$

$$u = 0 \quad \text{on } \Gamma_D \tag{13.3}$$

$$2\mu\sigma n = g \quad \text{on } \Gamma_N. \tag{13.4}$$

Here σ is stress and $\epsilon(u) = (\epsilon_{ij}(u))_{ij}$ with

$$\epsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j).$$

Alternatively the partial differential equation can be written as

$$-\mu\Delta u - \lambda\nabla \mathrm{div} u = f.$$

Variational formulation

Find $u \in H^1_D(\Omega)$ such that

$$a(u, v) = l(v) \tag{13.5}$$

where

$$a(u, v) = (\sigma(u), \epsilon(v)) = \int_\Omega 2\mu\epsilon(u) : \epsilon(v) + \lambda \mathrm{div} u \mathrm{div} v$$

and

$$l(v) = (f, v) + (g, v)_{\Gamma_N}$$

with

$$(f, v) = \int_\Omega \sum_{i=1}^{3} f_i v_i, \quad (g, v)_{\Gamma_N} = \int_{\Gamma_N} \sum_{i=1}^{3} g_i v_i.$$

Lemma 96 (Korn's inequality).

$$\|\epsilon(u)\|_{0,\Omega} \gtrsim |u|_{1,\Omega}.$$

The proof for the Korn's inequality is in the special case that $\Gamma_D \neq \Gamma$ and basically it follows from the following identity obtained by a simple integration by parts:

$$\|\epsilon(u)\|_{0,\Omega}^2 = \frac{1}{2}\|\nabla u\|_{0,\Omega}^2 + \|\mathrm{div} u\|_{0,\Omega}^2 \quad \forall\, u \in H_0^1(\Omega).$$

By Korn's inequality and Lax-Milgram Lemma, the variational problem is well-posed.

13.2 Finite element approximation and locking

Apparently, for any given parameters λ and μ, the finite element discretization for the linear elasticity is pretty much like that for the Poisson equations. For example, if $V_h \subset H_D^1(\Omega)$ is a piecewise linear finite element subspace, a finite element approximation $u_h \in V_h$ of the linear elasticity is then given by

$$a(u_h, v_h) = l(v_h) \quad \forall\, v_h \in H_{D,h}^1(\Omega).$$

Based on the following inequalities

$$\mu\|u_h\|_1^2 \lesssim a(u_h, u_h),$$

$$a(u_h, v_h) \lesssim (\mu + \lambda)\|u_h\|_1\|v_h\|_1,$$

u_h admits a standard a priori error estimate:

$$\|u - u_h\|_{1,\Omega} \lesssim \left(1 + \frac{\lambda}{\mu}\right)^{1/2} h|u|_{2,\Omega} \lesssim \left(1 + \frac{\lambda}{\mu}\right)^{1/2} h\|f\|_{0,\Omega}.$$

Here we emphasis that the dependence of Lamé constants for the constant in the above estimate.

There is a situation in which the above finite element discretization is not desirable, namely when $\lambda \to \infty$ or the so-called Poisson ration defined

$$\nu = \frac{\lambda}{2(\mu + \lambda)}$$

approaches to $1/2$.

It is easy to see that

$$\lim_{\nu \to 1/2} \mathrm{div} u = 0.$$

And for the finite element discretization mentioned above, we also have

$$\lim_{\nu \to 1/2} \mathrm{div} u_h = 0.$$

Therefore, as ν gets close to $1/2$, we expect the finite element solution u_h will get close to the following subspace

$$W_h = \{v_h \in V_h : \ \mathrm{div} v_h = 0\}.$$

For continuous piecewise linear element, one big problem is that the above finite element subspace is not very rich. For example $W_h \equiv \{0\}$ in some special case (see Exercise 1). Therefore, in this special case, we expect that $u_h \approx 0$ although the exact solution u may be different from zero. As a result, the finite element solution may fail to converge to u as the Poisson ratio gets close to $1/2$. This phenomenon is often known as "locking" in engineering literature.

Exercise 1. Prove that in the special that $\Omega = (0, 1)^2$ and $\Gamma_D = \partial\Omega$, $W_h \equiv \{0\}$ for a uniform triangulation on Ω.

Reduced integration

One remedy for the locking phenomenon is to resort to the following mixed formulations. Introduce an additional variable $p = \lambda\nabla u$. Then we can solve the following equations

$$\begin{cases} 2\mu(\epsilon(u), \epsilon(v)) + (\nabla \cdot v, p) = \langle f, v\rangle, & \forall v \in V, \\ (\nabla \cdot u, q) - \lambda^{-1}(p, q) = 0, & \forall q \in Q. \end{cases} \tag{13.6}$$

Here V and Q are continuous or discrete functions spaces. Since we are interested in large λ, we assume $\lambda \geq 1$.

The mixed formulation (13.6) is well-posed by standard arguments assuming stable Stokes FEM pairs are used. Namely, the inf-sup condition

$$\inf_{u,p} \sup_{v,q} \frac{L(u, p; v, q)}{(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0)} \geq \beta,$$

holds, where

$$L(u, p; v, q) = a(u, v) + b(u, q) + b(v, p) - \lambda^{-1}(p, q),$$

and $L(\cdot, \cdot)$ is bounded:

$$L(u, p; v, q) \leq C(\|u\|_1 + \|p\|_0)(\|v\|_1 + \|q\|_0).$$

Moreover, the constants β and C do not depend on λ. This results in error estimates that does not deteriorate as $\lambda \to +\infty$, i.e.

$$\|u - u_h\|_1 + \|\lambda\nabla \cdot u - p_h\|_0 \leq Ch(|u|_2 + \lambda|\nabla \cdot u|_1) \leq Ch\|f\|_0. \tag{13.7}$$

The constant C is independent of λ. Note that here we assume elliptic regularity.

In fact, the second equation in (13.6) can be written as follows

$$P_Q(\nabla \cdot u) - \lambda^{-1}p = 0,$$

where P_Q is the L^2 projection to the "pressure" space Q. By substitution, we get an equivalent problem:

$$2\mu(\epsilon(u), \epsilon(v)) + \lambda(P_Q\nabla \cdot u, \nabla \cdot v) = \langle f, v\rangle, \quad \forall v \in V. \tag{13.8}$$

The solution of this equaton u_h inherits the robust error estimates from (13.7); namely,

$$\|u - u_h\|_1 \leq Ch\|f\|_0. \tag{13.9}$$

The term $(P_Q\nabla \cdot u, \nabla \cdot v)$ in (13.8) is usually assembled by reducing integration order. It is sometimes referred to as "reduced intergration".

Kirchhoff plate model

In the special case of thin plate, three dimensional elasticity equation can be approximated by some two dimensional model. Let $\Omega = \tilde{\Omega} \times (-\epsilon, \epsilon)$. Define

$$H^2_D(\tilde{\Omega}) = \{v \in H^2(\tilde{\Omega}) : \ v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma_D\}. \tag{13.10}$$

In linear plate theory, the displacement can be decoupled in two parts, namely stretching and bending. In the so-called Kirchhoff plate model, the bending part of u can be approximated by $(-x_3\nabla w, w)$ for some function w defined on $\tilde{\Omega}$ which satisfies the following variational problem:

$$w \in H^2_D(\tilde{\Omega}), \quad (1 - \nu)(\nabla^2 w, \nabla^2 v)_{0,\tilde{\Omega}} + \nu(\Delta w, \Delta v)_{0,\tilde{\Omega}} = (\tilde{f}, v) \quad \forall\ v \in H^2_D(\tilde{\Omega}).$$

where

$$\tilde{f} = t^{-3}\frac{3(1-\nu^2)}{2E}\int_{-\epsilon}^{\epsilon}\Big(f_3 + x_3(\partial_1 f_1 + \partial_2 f_2)\Big)dx_3,$$

and E is the Young's module given by

$$E = \frac{\mu(2\mu+3\lambda)}{\mu+\lambda}.$$

The derivation of the above plate model is beyond our scope here. For interested readers, we refer to Ciarlet's book on elasticity. Our brief discussion would not give that much insight to this model, but it would at least gives some idea on where the equations come from.

13.3 Biharmonic equations

We shall have a more careful study of the Kirchhoff plate model mentioned above. With a slightly abuse of notation, we shall drop the $\sim$ in $\tilde{\Omega}$ and still use Ω denote a two dimensional domain. Let $H_D^2(\Omega)$ be defined as in (13.10). Thus we are interested in the following variational problem:

$$u \in H_D^2(\Omega), \quad a(u,v) = (f,v) \quad \forall\ v \in H_D^2(\Omega),$$

where

$$a(u,v) = (1-\nu)(\nabla^2 u, \nabla^2 v)_{0,\tilde{\Omega}} + \nu(\Delta u, \Delta v)_{0,\Omega}$$

It is easy to see that the bilinear form can be rewritten as

$$a(u,v) = (M, \nabla^2 v)_{0,\Omega}$$

where

$$M = (1-\nu)\nabla^2 u + \nu\Delta u\, I.$$

Now, integrating by parts twice,

$$\begin{aligned}(M,\nabla^2 v) &= -(\mathrm{div}M, \nabla v) + (Mn, \nabla v)\\ &= (\mathrm{div}\mathrm{div}M, \nabla v) - \langle n\cdot \mathrm{div}M, v\rangle + \langle Mn, \nabla v\rangle.\end{aligned}$$

We can write the last boundary term as

$$\begin{aligned}\langle Mn, \nabla v\rangle &= \langle n^T Mn, \frac{\partial v}{\partial n}\rangle + \langle s^T Mn, \frac{\partial v}{\partial s}\rangle\\ &= \langle n^T Mn, \frac{\partial v}{\partial n}\rangle - \langle \frac{\partial(s^T Mn)}{\partial s}, v\rangle\end{aligned}$$

where we have integrated by parts around the boundary in the last step.

Putting together all the above formula and also the simple identity that $\mathrm{div}\mathrm{div}M = \Delta^2 u$, we obtain that, for any $u, v \in H^2(\Omega)$,

$$a(u,v) = (\Delta^2 u, v) + \langle n^T Mn, \frac{\partial v}{\partial n}\rangle - \langle n\cdot \mathrm{div}M + \frac{\partial(s^T Mn)}{\partial s}, v\rangle.$$

We then deduce the strong form of the bending problem:

$$\Delta^2 u = f \quad \text{in } \Omega \tag{13.11}$$

$$u = 0, \ \frac{\partial u}{\partial n} = 0 \quad \text{on } \Gamma_D \tag{13.12}$$

$$n \cdot \mathrm{div} M + \frac{\partial(s^t M n)}{\partial s} = 0, \ n^T M n = 0 \quad \text{on } \Gamma_T. \tag{13.13}$$

We notice that (13.12) is an essential boundary condition while (13.13) is a natural boundary condition. Formally the boundary condition (13.13) can be written as

$$\frac{\partial(\Delta u)}{\partial n} + \frac{\partial^3 u}{\partial^2 s \partial n} - \kappa \frac{\partial^2 u}{\partial s^2} = 0, \tag{13.14}$$

where κ is the curvature function on the boundary.

In the application of finite element methods, we do not need to know much about the specifics of the natural boundary condition (13.13).

Other boundary condition and simply supported plate

In plate theory, it is also significant to consider other types of boundary conditions. As an example, let us briefly discuss the so-called simply supported plate which corresponds to the following variational problem:

$$u \in H^2(\Omega) \cap H_0^1(\Omega) \quad a(u, v) = (f, v) \quad \forall\, v \in H^2(\Omega) \cap H_0^1(\Omega).$$

We shall now briefly discuss an example of Babuska which demonstrates the boundary condition for certain boundary conditions need extraordinary caution when it comes for discretization. Consider now the simply supported plate in the unit disc Ω. Let $\Omega_m \subset \Omega$ be the $m-$polygonal approximation of Ω. Consider the simply supported plate in Ω_m and denote the corresponding solution by u_m. Babuska observed that, in general,

$$\lim_{m\to} u_m \neq u$$

although, apparently, $\lim_{m\to} \Omega_m = \Omega$. This phenomenon is known as *Babuska paradox.*

The key factor of verifying the Babuska paradox is that the natural boundary condition in the simply supported plate depends on the curvature of the boundary. While the curvature of the unit circle is identically one everywhere, the curvature of any polygon is zero except at the corners.

Exercise 2. Give a rigorous verification of Babuska paradox.

13.3.1 An example of conforming elements: Argyris triangle

Lemma 97. *Assume that $p \in P_5(T)$ satisfying*

$$D^\alpha p(a_i) = 0 \ (|\alpha| \le 2, 1 \le i \le 3), \quad \frac{\partial p}{\partial n} = 0 \ (1 \le i < j \le 3).$$

Then $p \equiv= 0$.

Proof. Consider the restriction of p on edge $e_1 = [a_2, a_3]$: $p_1(t) = p(a_2 + t(a_3 - a_2))$. By assumption

$$p_1(0) = p_1'(0) = p_1''(0) = p_1(1) = p_1'(1) = p_1''(1) = 0.$$

This means that 0 and 1 are both roots of p_1 of multiplicity 3. Thus $p_1 \equiv 0$ since $p_1 \in P_5(0, 1)$. A similar argument apparently also applies to other two edges, thus

$$p\Big|_{e_i} \equiv 0 \quad 1 \le i \le 3.$$

This fact implies that we can write

$$p = q\lambda_1\lambda_2\lambda_3 \text{ for some } q \in P_2(T).$$

Here $\lambda_i' s$ are the barycentric coordinates of T.

Let ∂_{e_i} denote the tangential derivative a long the edge e_i, then by assumption $(\partial_{e_1}\partial_{e_2}p(a_3) = 0$. But a direct computation shows that

$$\partial_{e_1}\partial_{e_2}p(a_3) = q(a_3)(\partial_{e_2}\lambda_1)(a_3)(\partial_{e_1}\lambda_2)(a_3)\lambda_3(a_3).$$

Thus we get $q_(a_3) = 0$. Similarly $q_(a_1) = q_(a_2) = 0$. Now another direction computation shows that

$$\frac{\partial p}{\partial n}(a_{23}) = q(a_3)\frac{\partial \lambda_1}{\partial n})(a_3)\lambda_2(a_3)\lambda_3(a_3).$$

we conclude that $q(a_{23}) = 0$ since by assumption $\frac{\partial p}{\partial n}(a_{23}) = 0$, but apparently $\frac{\partial \lambda_1}{\partial n})(a_3)\lambda_2(a_3)\lambda_3(a_3) \neq 0$. Similarly $q(a_{12}) = q(a_{31}) = 0$. In summary

$$q(a_k) = 0 (1 \le k \le 3) \quad q(a_{ij}) = 0 (1 \le i < j \le 3) = 0.$$

This implies that $q \equiv 0$ since $q \in P_2(\Omega)$. □

13.4 Mixed Finite Element Method: Symmetric Tensor

In this section, we will design the mixed finite element methods in linear elasticity with symmetric stress approximations. These finite element spaces are defined with respect to an arbitrary simplicial triangulation of the domain, and can be regarded as extensions to any dimension.

13.4.1 Finite elements for symmetric tensors

We consider mixed finite element methods of first order systems with symmetric tensors: Find $(\sigma, u) \in \Sigma \times V := H(\text{div}, \Omega; \mathbb{S}) \times L^2(\Omega; \mathbb{R}^n)$, such that

$$(13.15) \qquad \begin{cases} (A\sigma, \tau) + (\text{div}\tau, u) = 0 & \text{for all } \tau \in \Sigma, \\ \qquad (\text{div}\sigma, v) = (f, v) & \text{for all } v \in V. \end{cases}$$

Here the symmetric tensor space for the stress Σ is defined by

$$(13.16) \qquad H(\text{div}, \Omega; \mathbb{S}) := \left\{ \tau = \begin{pmatrix} \tau_{11} & \cdots & \tau_{1n} \\ \vdots & \vdots & \vdots \\ \tau_{n1} & \cdots & \tau_{nn} \end{pmatrix} \in H(\text{div}, \Omega; \mathbb{R}^{n\times n}),\ \tau^T = \tau \right\},$$

and the space for the vector displacement V is

$$(13.17) \qquad L^2(\Omega; \mathbb{R}^n) := \left\{ (u_1, \cdots, u_n)^T,\ u_i \in L^2(\Omega), i = 1, \cdots, n \right\}.$$

Here denotes by $H^k(T, X)$ the Sobolev space consisting of functions with domain $T \subset \mathbb{R}^n$, taking values in the finite-dimensional vector space X, and with all derivatives of order at most k square-integrable. For our purposes, the range space X will be either $\mathbb{S}$, $\mathbb{R}^n$, or $\mathbb{R}$. Let $\|\cdot\|_{k,T}$ be the norm of $H^k(T)$. $\mathbb{S}$ denotes the space of symmetric tensors, $H(\text{div}, T, \mathbb{S})$ consists of square-integrable symmetric matrix fields with square-integrable divergence. The H(div) norm is defined by

$$\|\tau\|^2_{H(\mathrm{div},T)} := \|\tau\|^2_{0,T} + \|\mathrm{div}\tau\|^2_{0,T}.$$

$L^2(T,\mathbb{R}^n)$ is the space of vector-valued functions which are square-integrable. Here, the compliance tensor $A = A(x) \in L^\infty(\mathbb{S};\mathbb{S})$, characterizing the properties of the material, is bounded and symmetric positive definite uniformly for $x \in \Omega$.

Suppose that the domain Ω is subdivided by a family of shape regular simplicial grids $\mathcal{T}_h$ (with the grid size h). We introduce the finite element space of order $k \geq 1$ on $\mathcal{T}_h$.

$$\Sigma_{k,h} := \left\{\sigma \in H(\mathrm{div},\Omega;\mathbb{S}),\ \sigma|_K \in P_k(K;\mathbb{S}) \ \text{ for all } K \in \mathcal{T}_h\right\}, \tag{13.18}$$

where $P_k(K;X)$ denotes the space of polynomials of degree $\leq k$, taking value in the space X.

A new basis of the symmetric matrices

Let $\mathbf{x}_0, \cdots, \mathbf{x}_n$ be the vertices of simplex K. The referencing mapping is then

$$\mathbf{x} := F_K(\hat{\mathbf{x}}) = \mathbf{x}_0 + (\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0)\,\hat{\mathbf{x}},$$

mapping the reference tetrahedron $\hat{K} := \{0 \leq \hat{x}_1, \cdots, \hat{x}_n, 1 - \sum_{i=1}^n \hat{x}_i \leq 1\}$ to K. Then the inverse mapping is

$$\hat{\mathbf{x}} := \begin{pmatrix} \nu_1^T \\ \vdots \\ \nu_n^T \end{pmatrix} (\mathbf{x} - \mathbf{x}_0), \tag{13.19}$$

where

$$\begin{pmatrix} \nu_1^T \\ \vdots \\ \nu_n^T \end{pmatrix} = (\mathbf{x}_1 - \mathbf{x}_0, \cdots, \mathbf{x}_n - \mathbf{x}_0)^{-1}. \tag{13.20}$$

By (13.19), these normal vectors are coefficients of the barycentric variables:

$$\begin{aligned} \lambda_1(\mathbf{x}) : &= \nu_1 \cdot (\mathbf{x} - \mathbf{x}_0), \\ &\vdots \\ \lambda_n(\mathbf{x}) : &= \nu_n \cdot (\mathbf{x} - \mathbf{x}_0), \\ \lambda_0(\mathbf{x}) : &= 1 - \sum_{i=1}^n \lambda_i. \end{aligned}$$

For any edge $\mathbf{x}_i\mathbf{x}_j$ of element K, $i \neq j$, let $\mathbf{t}_{i,j}$ denote associated tangent vectors, which allow for us to introduce the following symmetric matrices of rank one

$$T_{i,j} := \mathbf{t}_{i,j}\mathbf{t}_{i,j}^T, 0 \leq i < j \leq n. \tag{13.21}$$

For these matrices of rank one, we have the following important result.

Lemma 98. *The* $\frac{(n+1)n}{2}$ *symmetric tensors* $T_{i,j}$ *in* (13.21) *are linearly independent, and form a basis of* $\mathbb{S}$.

Proof. Each matrix $T_{i,j} = \mathbf{t}_{i,j}\mathbf{t}_{i,j}^T$ is a positive semi-definite matrix, on a simplex K. We would show that the constants $c_{i,j}$ are all equal to zero in

$$\tau = \sum_{0\le i<j\le n} c_{i,j}T_{i,j} = 0.$$

Let ν_0 be the normal vector to the $n-1$ dimensional simplex $\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n$. This leads to

$$\nu_0^T\mathbf{t}_{i,j} = 0, 1 \le i < j \le n, \tag{13.22}$$

and

$$\nu_0^T\mathbf{t}_{0,j} \neq 0, 1 \le j \le n. \tag{13.23}$$

This gives

$$\begin{aligned}\nu_0^T\tau = \nu_0^T\sum_{0\le i<j\le n} c_{i,j}\mathbf{t}_{i,j}\mathbf{t}_{i,j}^T &= \nu_0^T\sum_{1\le j\le n} c_{0,j}\mathbf{t}_{0,j}\mathbf{t}_{0,j}^T\\ &= \sum_{1\le j\le n}\tilde{c}_{0,j}\mathbf{t}_{0,j}^T = 0,\end{aligned} \tag{13.24}$$

where $\tilde{c}_{0,j} = c_{0,j}\nu_0^T\mathbf{t}_{0,j}$. Since $\mathbf{t}_{0,j}$, $1 \le j \le n$, are linearly independent, this yields

$$\tilde{c}_{0,j} = 0, 1 \le j \le n. \tag{13.25}$$

This and (13.23) yield

$$c_{0,j} = 0, 1 \le j \le n. \tag{13.26}$$

A similar argument by using ν_i, $i \neq 0$, proves the desired result. □

The bubble–function space

With these symmetric matrices $T_{i,j}$ of rank one, we define a $H(\mathrm{div}, K; \mathbb{S})$ bubble function space

$$\Sigma_{K,k,b} := \sum_{0\le i<j\le n}\lambda_i\lambda_j P_{k-2}(K;\mathbb{R})T_{i,j}. \tag{13.27}$$

Then we have the following *crucial* structure for $\Sigma_{k,h}$:

$$\begin{aligned}\Sigma_{k,h} := \Big\{ &\sigma \in H(\mathrm{div},\Omega,\mathbb{S}), \sigma = \sigma_c + \sigma_b,\ \sigma_c \in H^1(\Omega,\mathbb{S}),\\ &\sigma_c|_K \in P_k(K,\mathbb{S}),\ \sigma_b|_K \in \Sigma_{K,k,b}, \forall K \in \mathcal{T}_h\Big\},\end{aligned} \tag{13.28}$$

which is a $H(\mathrm{div})$ bubble enrichment of the H^1 space $\widetilde{\Sigma}_{k,h} := \Sigma_{k,h} \cap H^1(\Omega;\mathbb{S})$ of $\Sigma_{k,h}$. This generalizes the results of [**?**, **?**] for both two and three dimensions to the general case in any space dimension. Such a structure has already enabled us to write down directly the basis of $\Sigma_{k,h}$; see [**?**, **?**] for more details in both two and three dimensions. Next we plan, as it has been done for most of usual finite element methods in the literature, to define a set of local degrees of freedom of shape function spaces $P_k(K,\mathbb{S})$ on each element. To this end, we define the full $H(\mathrm{div}, K; \mathbb{S})$ bubble function space consisting of polynomials of degree $\le k$

$$\Sigma_{\partial K,k,0} := \{\tau \in H(\mathrm{div}, K;\mathbb{S}) \cap P_k(K;\mathbb{S}), \tau\nu|_{\partial K} = 0\}. \tag{13.29}$$

Here ν is the normal vector of ∂K.

Lemma 99. *It holds that*

$$\Sigma_{K,k,b} = \Sigma_{\partial K,k,0}. \tag{13.30}$$

Proof. Consider a function $\tau \in \lambda_i\lambda_j P_{k-2}(K;\mathbb{R})T_{i,j}$, $0 \le i < j \le n$. Note that τ vanishes on the $n-1$ dimensional simplices

$$\triangle_{n-1}\mathbf{x}_0\cdots\mathbf{x}_{i-1}\mathbf{x}_{i+1}\cdots\mathbf{x}_j\cdots\mathbf{x}_n,$$

$$\triangle_{n-1}\mathbf{x}_0\cdots\mathbf{x}_i\cdots\mathbf{x}_{j-1}\mathbf{x}_{j+1}\cdots\mathbf{x}_n.$$

For any $n-1$ dimensional simplex which takes edge $\mathbf{x}_i\mathbf{x}_j$, its normal vector, say ν, is perpendicular to the tangent vector $\mathbf{t}_{i,j}$ of edge $\mathbf{x}_i\mathbf{x}_j$, which implies that $\tau\nu = 0$ on such a $n-1$ dimensional simplex and consequently $\tau \in \Sigma_{\partial K,k,0}$. Hence

$$\Sigma_{K,k,b} \subset \Sigma_{\partial K,k,0}. \tag{13.31}$$

Next we show the converse of (13.31). Given $\tau \in \Sigma_{\partial K,k,0}$, the boundary condition $\tau\nu|_{\partial K} = 0$ indicates that τ vanishes at all the vertices of K. Let $\mathbb{N}_b$ denote all the nodes except the vertices of K for the space $P_k(K;\mathbb{R})$. Given $\mathbb{P}_\ell \in \mathbb{N}_b$, let $\varphi_\ell \in P_k(K;\mathbb{R})$ denote the usual associated nodal Lagrange basis function, namely, $\varphi_\ell(\mathbb{P}_\ell) = 1$ and φ_ℓ vanishes at all the other nodes for the space $P_k(K;\mathbb{R})$. It follows from Lemma 98 that

$$\tau = \sum_{0\le i<j\le n} T_{i,j}\Big(\sum_{\mathbb{P}_\ell\in\mathbb{N}_b} c_{\ell,i,j}\varphi_\ell\Big). \tag{13.32}$$

Note that φ_ℓ has a homogeneous expression by $\lambda_0,\cdots,\lambda_n$. Therefore, we have

$$\sum_{\mathbb{P}_\ell\in\mathbb{N}_b} c_{\ell,i,j}\varphi_\ell = \sum_{m_0+m_1+\cdots+m_n=k} c_{(ij),m_0,m_1,\cdots,m_n}\lambda_0^{m_0}\cdots\lambda_n^{m_n}. \tag{13.33}$$

We claim that $\sum_{\mathbb{P}_\ell\in\mathbb{N}_b} c_{\ell,i,j}\varphi_\ell$ has a factor $\lambda_i\lambda_j$, namely,

$$\sum_{\mathbb{P}_\ell\in\mathbb{N}_b} c_{\ell,i,j}\varphi_\ell = \lambda_i\lambda_j \sum_{m_0'+m_1'+\cdots+m_n'=k-2} c'_{(ij),m_0',m_1',\cdots,m_n'}\lambda_0^{m_0'}\cdots\lambda_n^{m_n'}. \tag{13.34}$$

Without loss of generality, we consider the case where $i = 0$ and $j = 1$. Suppose that there is a term $f_1T_{0,1}$ such that f_1 is a polynomial of degree $\le k$ and does not contain a factor λ_0. Next we shall show that $f_1 = 0$. In fact, all the terms of (13.32) which do not contain the factor λ_0 and whose normal components (namely $T_{0,j}\nu_0 \ne 0$, ν_0 is the normal vector of $\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n$) do not vanish on the $n-1$ dimensional simplex $\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n$ can be expressed as

$$\sum_{j=1}^{n} f_jT_{0,j}, \tag{13.35}$$

where f_j, $j = 1,\cdots,n$, are polynomials of degree $\le k$. Since f_j do not contain the factor λ_0, it is of the form

$$f_j = \sum_{r_1+\cdots+r_n=k} c_{j,r_1,\cdots,r_n}\lambda_1^{r_1}\cdots\lambda_n^{r_n}. \tag{13.36}$$

Since $\tau\nu_0 = 0$ on the $n-1$ dimensional simplex $\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n$,

$$\sum_{j=1}^{n}(\mathbf{t}_{0,j}^T\nu_0)f_j\mathbf{t}_{0,j}\Big|_{\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n} = 0. \tag{13.37}$$

Since, for $j = 1, \cdots, n$, $\mathbf{t}_{0,j}^T \nu_0 \neq 0$, and $\mathbf{t}_{0,j}$ are linearly independent, this leads to

$$f_j|_{\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n} \equiv 0. \tag{13.38}$$

Note that $\lambda_1^{r_1}\cdots\lambda_n^{r_n}|_{\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n}$, $\sum_{i=1}^n r_i = k$, form a basis of $P_k(\triangle_{n-1}\mathbf{x}_1\cdots\mathbf{x}_n;\mathbb{R})$. This and the above equation show that

$$c_{j,r_1,\cdots,r_n} = 0. \tag{13.39}$$

This, in turn, implies that

$$f_j \equiv 0. \tag{13.40}$$

Therefore $f_1 = 0$ which implies that all the terms on the right hand side of (13.33) has a factor λ_0. A similar argument shows that all the terms on the right hand side of (13.33) has a factor λ_1. Hence

$$\tau \in \Sigma_{K,k,b}. \tag{13.41}$$

This completes the proof. □

Degrees of freedom

Before we define the degrees of freedom, we need a classical result and its variant.

Lemma 100. *It holds the following Chu-Vandermonde combinatorial identity and its variant*

$$\sum_{\ell=0}^n C_{n+1}^{\ell+1}C_{k-1}^{\ell} = \sum_{\ell=0}^n C_{n+1}^{n-\ell}C_{k-1}^{\ell} = C_{n+k}^n, \tag{13.42}$$

$$\sum_{\ell=0}^n C_{n+1}^{\ell+1}C_{k-1}^{\ell}C_{\ell+1}^2 = \frac{(n+1)n}{2}C_{n+k-2}^n, \tag{13.43}$$

where the combinatorial number $C_n^m = \frac{n\cdots(n-m+1)}{m\cdots 1}$ *for* $n \geq m$ *and* $C_n^m = 0$ *for* $n < m$.

Proof. The identity (13.42) is classical, and (13.43) is its variant. For readers' convenience, we sketch the proof for them. It follows from the well-known binomial theorem that

$$(1+t)^{n+1}(1+t)^{k-1} = (1+t)^{n+k} = \sum_{m=0}^{n+k} C_{n+k}^m t^m.$$

On the other hand, we have

$$(1+t)^{n+1}(1+t)^{k-1} = \sum_{m_1=0}^{n+1} C_{n+1}^{m_1}t^{m_1}\sum_{m_2=0}^{k-1} C_{k-1}^{m_2}t^{m_2}.$$

A combination of these two equations leads to

$$C_{n+k}^n = \sum_{m_1+m_2=n} C_{n+1}^{m_1}C_{k-1}^{m_2} = \sum_{\ell=0}^n C_{n+1}^{n-\ell}C_{k-1}^{\ell},$$

which proves (13.42). To prove (13.43), we consider

$$\sum_{\ell=0}^{n} C_{n+1}^{\ell+1} C_{k-1}^{\ell} \frac{(\ell+1)\ell}{(n+1)n} = \sum_{\ell=0}^{n} C_{n-1}^{\ell-1} C_{k-1}^{\ell} = \sum_{\ell=0}^{n} C_{n-1}^{n-\ell} C_{k-1}^{\ell} = C_{n+k-2}^{n}.$$

□

Theorem 98. *A matrix field $\tau \in P_k(K;\mathbb{S})$ can be uniquely determined by the degrees of from (1) and (2)*

1. *For each ℓ dimensional simplex $\triangle_\ell$ of K, $0 \le \ell \le n-1$, with ℓ linearly independent tangential vectors $\mathbf{t}_1, \cdots, \mathbf{t}_\ell$, and $n-\ell$ linearly independent normal vectors $\nu_1, \cdots, \nu_{n-\ell}$, the mean moments of degree at most $k-\ell-1$ over $\triangle_\ell$, of $\mathbf{t}_l^T \tau \nu_i$, $\nu_i^T \tau \nu_j$, $l = 1, \cdots, \ell$, $i, j = 1, \cdots, n-\ell$, $(C_{n+1-\ell}^2 + \ell(n-\ell))C_{k-1}^{\ell} = \frac{(n-\ell)(n+\ell+1)}{2} C_{k-1}^{\ell}$ degrees of freedom for each $\triangle_\ell$;*
2. *the values $\int_K \tau : \theta d\mathbf{x}$ for any $\theta \in \Sigma_{K,k,b}$, $\frac{(n+1)n}{2} C_{n+k-2}^{n}$ degrees of freedom.*

Proof. We assume that all degrees of freedom vanish and show that $\tau = 0$. Note that the mean moment becomes the value of τ for a 0 dimensional simplex $\triangle_0$, namely, a vertex, of K. The first set of degrees of freedom imply that $\tau\nu = 0$ on ∂K. Then the second set of degrees of freedom and Lemma 99 show $\tau = 0$. Since the number of degrees of freedom in the second set follows immediately from Lemma 98, we only need to prove the number of degrees of freedom in the first set. The number of $\mathbf{t}_l^T \tau \nu_i$, $l = 1, \cdots, \ell$, $i = 1, \cdots, n-\ell$, is

$$\ell(n-\ell),$$

while, by symmetry, the number of $\nu_i^T \tau \nu_j$, $i, j = 1, \cdots, n-\ell$, reads

$$\frac{(n-\ell)(n-\ell+1)}{2}.$$

The number of the mean moments of degree at most $k-\ell-1$ over $\triangle_\ell$ is C_{k-1}^{ℓ}. These imply the number of degrees of freedom in the first set is

$$(\ell(n-\ell) + \frac{(n-\ell)(n-\ell+1)}{2})C_{k-1}^{\ell} = \frac{(n-\ell)(n+\ell+1)}{2} C_{k-1}^{\ell}.$$

Hence the sum of degrees of freedom in both sets reads

$$\begin{aligned} &\sum_{\ell=0}^{n-1} C_{n+1}^{\ell+1} \frac{(n-\ell)(n+\ell+1)}{2} C_{k-1}^{\ell} + \frac{(n+1)n}{2} C_{n+k-2}^{n} \\ &= \frac{(n+1)n}{2} \sum_{\ell=0}^{n-1} C_{n+1}^{\ell+1} C_{k-1}^{\ell} - \sum_{\ell=0}^{n-1} \frac{\ell(\ell+1)}{2} C_{n+1}^{\ell+1} C_{k-1}^{\ell} + \frac{(n+1)n}{2} C_{n+k-2}^{n} \\ &= \frac{(n+1)n}{2} \sum_{\ell=0}^{n} C_{n+1}^{\ell+1} C_{k-1}^{\ell} - \sum_{\ell=0}^{n} \frac{\ell(\ell+1)}{2} C_{n+1}^{\ell+1} C_{k-1}^{\ell} + \frac{(n+1)n}{2} C_{n+k-2}^{n}. \end{aligned}$$

Then it follows from the Chu-Vandermonde combinatorial identity (13.42) and its variant (13.43) that it is equal to $\frac{n(n+1)}{2} C_{n+k}^{n}$ the dimension of $P_k(K;\mathbb{S})$. □

Remark 28. It follows from Theorem 98 that, for any dimension, if $k = 1$, $\Sigma_{k,h}$ becomes a H^1 conforming approximation of $\Sigma := H(\mathrm{div}, \Omega; \mathbb{S})$. For one dimensional case with $n = 1$, for any k, $\Sigma_{k,h}$ becomes the usual H^1 finite element space of degree k.

The divergence space of the bubble function space

Before ending this section, we prove an important result concerning the divergence space of the bubble function space. To this end, we introduce the following rigid motion space on each element K.

$$R(K) := \{v \in H^1(K;\mathbb{R}^n), (\nabla v + \nabla v^T)/2 = 0\}. \tag{13.44}$$

It follows from the definition that $R(K)$ is a subspace of $P_1(K;\mathbb{R}^n)$. For $n = 1$, $R(K)$ is the constant function space over K. The dimension of $R(K)$ is $\frac{n(n+1)}{2}$. This allows for defining the orthogonal complement space of $R(K)$ with respect to $P_{k-1}(K;\mathbb{R}^n)$ by

$$R^{\perp}(K) := \{v \in P_{k-1}(K;\mathbb{R}^n), (v,w)_K = 0 \text{ for any } w \in R(K)\}, \tag{13.45}$$

where the inner product $(v,w)_K$ over K reads $(v,w)_K = \int_K v \cdot w d\mathbf{x}$.

Theorem 99. *For any $K \in \mathcal{T}_h$, it holds that*

$$\operatorname{div} \Sigma_{K,k,b} = R^{\perp}(K). \tag{13.46}$$

Proof. For any $\tau \in \Sigma_{K,k,b}$, an integration by parts yields

$$\int_K \operatorname{div} \tau \cdot w d\mathbf{x} = 0 \text{ for any } w \in R(K).$$

This implies that

$$\operatorname{div} \Sigma_{K,k,b} \subset R^{\perp}(K). \tag{13.47}$$

Next we show the converse. In fact, if $\operatorname{div} \Sigma_{K,k,b} \neq R^{\perp}(K)$, there is a nonzero $v \in R^{\perp}(K)$ such that

$$\int_K \operatorname{div} \tau \cdot v \, d\mathbf{x} = 0 \quad \forall \tau \in \Sigma_{K,k,b}.$$

By integration by parts, for $\tau \in \Sigma_{K,k,b}$, we have

$$\int_K \operatorname{div} \tau \cdot v d\mathbf{x} = -\int_K \tau : \epsilon(v) d\mathbf{x} = 0, \tag{13.48}$$

where $\epsilon(v)$ is the symmetric gradient, $(\nabla v + \nabla^T v)/2$.

By Lemma 98, $T_{i,j}$, $0 \le i < j \le n$ defined in (13.21), form a basis of the space of symmetric matrices in $\mathbb{R}^{n\times n}$. Then there exists an associated dual basis, say $M_{i,j}$, $0 \le i < j \le n$, such that

$$T_{i,j} : M_{k,l} = \delta_{i,k}\delta_{j,l}, 0 \le i < j \le n, 0 \le k < l \le n. \tag{13.49}$$

Here the inner product of two matrices $A = (a_{ij})_{i,j=1}^n$ and $B = (b_{ij})_{i,j=1}^n$ is defined as

$$A : B = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}.$$

As $\epsilon(v) \in P_{k-2}(K;\mathbb{S})$, it follows that there exist $q_{i,j} \in P_{k-2}(K;\mathbb{R})$, $0 \le i < j \le n$, such that

$$\epsilon(v) = \sum_{0\le i<j\le n} q_{i,j} M_{i,j}. \tag{13.50}$$

Selecting $\tau = \sum_{0\le i<j\le n} \lambda_i\lambda_j q_{i,j} T_{i,j} \in \Sigma_{K,k,b}$, we have,

$$0 = \int_K \tau : \epsilon(v) d\mathbf{x} = \sum_{0\le i<j\le n} \int_K \lambda_i \lambda_j q_{i,j}^2(\mathbf{x}) d\mathbf{x}.$$

As $\lambda_i\lambda_j > 0$ on K, we conclude that $q_{i,j} \equiv 0$, which implies that v is a rigid motion. This contradicts with $v \in R^{\perp}(K)$. Hence $R^{\perp}(K) \subset \operatorname{div} \Sigma_{K,k,b}$, which completes the proof. □

13.4.2 Mixed methods of first order systems with symmetric tensors

13.4.3 Mixed methods

We propose to use the space $\Sigma_{k,h}$, with $k \geq n+1$, defined in (13.18) to approximate Σ. In order get a stable pair of spaces, we take the discrete displacement space as the full C^{-1}-P_{k-1} space

$$V_{k,h} := \{v \in L^2(\Omega;\mathbb{R}^n), v|_K \in P_{k-1}(K;\mathbb{R}^n) \text{ for all } K \in \mathcal{T}_h\}. \tag{13.51}$$

follows from the definition of $V_{k,h}$ (P_{k-1} polynomials) and $\Sigma_{k,h}$ (P_k polynomials) that

$$\operatorname{div} \Sigma_{k,h} \subset V_{k,h}.$$

This, in turn, leads to a strong divergence-free space:

$$\begin{aligned} Z_h &:= \{\tau_h \in \Sigma_{k,h} \mid (\operatorname{div} \tau_h, v) = 0 \quad \text{for all } v \in V_{k,h}\} \\ &= \{\tau_h \in \Sigma_{k,h} \mid \operatorname{div} \tau_h = 0 \text{ pointwise }\}. \end{aligned} \tag{13.52}$$

The mixed finite element approximation of Problem (1.1) reads: Find $(\sigma_h,\ u_h) \in \Sigma_{k,h} \times V_{k,h}$ such that

$$\begin{cases} (A\sigma_h, \tau) + (\operatorname{div}\tau, u_h) = 0 & \text{for all } \tau \in \Sigma_{k,h}, \\ \qquad (\operatorname{div} \sigma_h, v) = (f, v) & \text{for all } v \in V_{k,h}. \end{cases} \tag{13.53}$$

Stability analysis and error estimates

The convergence of the finite element solutions follows the stability and the standard approximation property. So we consider first the well-posedness of the discrete problem (13.53). By the standard theory, we only need to prove the following two conditions, based on their counterpart at the continuous level.

1. K-ellipticity. There exists a constant $C > 0$, independent of the meshsize h such that

   $$(A\tau, \tau) \geq C\|\tau\|^2_{H(\mathrm{div})} \quad \text{for all } \tau \in Z_h, \tag{13.54}$$

   where Z_h is the divergence-free space defined in (13.52).
2. Discrete B-B condition. There exists a positive constant $C > 0$ independent of the meshsize h, such that

   $$\inf_{0\neq v\in V_{k,h}} \sup_{0\neq\tau\in\Sigma_{k,h}} \frac{(\operatorname{div}\tau, v)}{\|\tau\|_{H(\mathrm{div})}\|v\|_0} \geq C. \tag{13.55}$$

It follows from $\operatorname{div} \Sigma_{k,h} \subset V_{k,h}$ that $\operatorname{div} \tau = 0$ for any $\tau \in Z_h$. This implies the above K-ellipticity condition (13.54). It remains to show the discrete B-B condition (14.36), in the following two lemmas.

We recall the subspace $\widetilde{\Sigma}_{k,h} := \Sigma_{k,h} \cap H^1(\Omega;\mathbb{S})$ of $\Sigma_{k,h}$. For $\tau \in \widetilde{\Sigma}_{k,h}$, the degrees of freedom on any element K are: for each ℓ dimensional simplex $\triangle_\ell$ of K, $0 \leq \ell \leq n$, the mean moments of degree at most $k-\ell-1$ over $\triangle_\ell$, of τ. A standard argument is able to prove that these degrees of freedom are unisolvent.

Lemma 101. *For any $v_h \in V_{k,h}$, there is a $\tau_h \in \widetilde{\Sigma}_{k,h}$ such that, for all polynomial $p \in R(K)$, $K \in \mathcal{T}_h$,*

$$\int_K (\operatorname{div} \tau_h - v_h) \cdot p \, d\mathbf{x} = 0 \quad \text{and} \quad \|\tau_h\|_{H(\mathrm{div})} \leq C\|v_h\|_0. \tag{13.56}$$

Proof. Let $v_h \in V_{k,h}$. By the stability of the continuous formulation, cf. [?] for two dimensional case, there is a $\tau \in H^1(\Omega;\mathbb{S})$ such that,

$$\operatorname{div}\tau = v_h \quad \text{and} \quad \|\tau\|_1 \le C\|v_h\|_0.$$

In this paper, we only consider the domain such that the above stability holds. We refer interested authors to [?] for the classical result which states it is true for Lipschitz domains in $\mathbb{R}^n$; see [?] for more refined results. First let I_h be a Scott-Zhang [?] interpolation operator such that

$$\|\tau - I_h\tau\|_0 + h\|\nabla I_h\tau\|_0 \le Ch\|\nabla\tau\|_0. \tag{13.57}$$

Since $k \ge n+1$, $k-(n-1)-1 \ge 1$, for each $n-1$ dimensional simplex $\triangle_{n-1}$ of K, there are at least n bubble functions on $\triangle_{n-1}$ for each component of τ. In fact let $\mathbb{T}_{ij}$, $1 \le i < j \le n$ be the canonical basis of the space $\mathbb{S}$. There are C_{k-1}^{k-n} Lagrange basis functions $\varphi_\ell \in \{p \in H^1(\Omega;\mathbb{R}), p|_K \in P_k(K;\mathbb{R}), \text{for any } K \in \mathcal{T}_h\}$, $\ell = 1,\cdots,C_{k-1}^{k-n}$, such that φ_ℓ vanish on $\partial(K^+ \cup K^-)$, where K^+ and K^- are two elements that share the common $n-1$ dimensional simplex $\triangle_{n-1}$. Then $\varphi_\ell\mathbb{T}_{ij}$, $1 \le i < j \le n$, $\ell = 1,\cdots,C_{k-1}^{k-n}$, are matrix–valued bubble functions,which are linearly independent. These bubble functions allow for defining a correction $\delta_h \in \widetilde{\Sigma}_{k,h}$ such that

$$\int_{\triangle_{n-1}} \delta_h\nu\cdot p\,d\mathbf{s} = \int_{\triangle_{n-1}} (\tau - I_h\tau)\nu\cdot p\,d\mathbf{s} \text{ for any } p \in R(K)|_{\triangle_{n-1}}. \tag{13.58}$$

Finally we take

$$\tau_h = I_h\tau + \delta_h. \tag{13.59}$$

We get a partial-divergence matching property of τ_h: for any $p \in R(K)$, as the symmetric gradient $\epsilon(p) = 0$,

$$\begin{aligned}\int_K(\operatorname{div}\tau_h - v_h)\cdot p\,d\mathbf{x} &= \int_K(\operatorname{div}\tau_h - \operatorname{div}\tau)\cdot p\,d\mathbf{x}\\ &= \int_{\partial K}(\tau_h - \tau)\nu\cdot p\,d\mathbf{s} = 0.\end{aligned}$$

It remain to show the stability estimate. It is standard to use a scaling argument and the trace theory to show that

$$\|\delta_h\|_0 + h\|\nabla\delta_h\|_0 \le C(\|\tau - I_h\tau\|_0 + h\|\nabla(\tau - I_h\tau)\|_0).$$

Then the stability estimate in (13.56) follows from (13.57) and the triangle inequality. □

We are in the position to show the well-posedness of the discrete problem.

Theorem 100. *For the discrete problem (13.53), the K-ellipticity* (13.54) *and the discrete B-B condition* (14.36) *hold uniformly. Consequently, the discrete mixed problem* (13.53) *has a unique solution* $(\sigma_h, u_h) \in \Sigma_{k,h} \times V_{k,h}$.

Proof. The K-ellipticity immediately follows from the fact that $\operatorname{div}\Sigma_{k,h} \subset V_{k,h}$. To prove the discrete B-B condition (14.36), for any $v_h \in V_{k,h}$, it follows from Lemma 101 that there exists a $\tau_1 \in \Sigma_{k,h}$ such that, for any polynomial $p \in R(K)$,

$$\int_K(\operatorname{div}\tau_1 - v_h)\cdot p d\mathbf{x} = 0 \quad \text{and} \quad \|\tau_1\|_{H(\mathrm{div})} \le C\|v_h\|_0. \tag{13.60}$$

Then it follows from Theorem 99 that there is a $\tau_2 \in \Sigma_{k,h}$ such that $\tau_2|_K \in \Sigma_{K,k,b}$ and

$$\operatorname{div}\tau_2 = v_h - \operatorname{div}\tau_1, \|\tau_2\|_0 = \min\{\|\tau\|_0, \operatorname{div}\tau = v_h - \operatorname{div}\tau_1, \tau \in \Sigma_{K,k,b}\} \tag{13.61}$$

It follows from the definition of τ_2 that $\|\operatorname{div}\tau_2\|_0$ defines a norm for it. Then, a scaling argument proves

$$\|\tau_2\|_{H(\mathrm{div})} \le C\|\operatorname{div}\tau_1 - v_h\|_0. \tag{13.62}$$

Let $\tau = \tau_1 + \tau_2$. This implies that

$$\operatorname{div}\tau = v_h \text{ and } \|\tau\|_{H(\mathrm{div})} \le C\|v_h\|_0, \tag{13.63}$$

this proves the discrete B-B condition (14.36). □

Theorem 101. *Let* $(\sigma, u) \in \Sigma \times V$ *be the exact solution of problem* (20.42) *and* $(\tau_h, u_h) \in \Sigma_{k,h} \times V_{k,h}$ *the finite element solution of* (13.53)*. Then, for* $k \ge n+1$*,*

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_0 \le Ch^k(\|\sigma\|_{k+1} + \|u\|_k). \tag{13.64}$$

Proof. stability of the elements and the standard theory of mixed element methods [**?**, 10] give the following quasioptimal error estimate immediately

$$\|\sigma - \sigma_h\|_{H(\mathrm{div})} + \|u - u_h\|_0 \le C \inf_{\tau_h\in\Sigma_{k,h}, v_h\in V_{k,h}} \left(\|\sigma - \tau_h\|_{H(\mathrm{div})} + \|u - v_h\|_0\right). \tag{13.65}$$

Let P_h denote the local L^2 projection operator, or element-wise interpolation operator, from V to $V_{k,h}$, satisfying the error estimate

$$\|v - P_h v\|_0 \le Ch^k\|v\|_k \text{ for any } v \in H^k(\Omega; \mathbb{R}^n). \tag{13.66}$$

Choosing $\tau_h = I_h\sigma \in \Sigma_{k,h}$ where I_h is defined in (13.57) as I_h preserves symmetric P_k functions locally,

$$\|\sigma - \tau_h\|_0 + h|\sigma - \tau_h|_{H(\mathrm{div})} \le Ch^{k+1}\|\sigma\|_{k+1}. \tag{13.67}$$

Let $v_h = P_h v$ and $\tau_h = I_h\sigma$ in (13.65), by (13.66) and (13.67), we obtain (13.64). □

Remark 29. To prove an optimal error estimate for the stress in the L^2 norm, we can follow the idea from [**?**] to use a mesh dependent norm technique. In particular, this will lead to

$$\|\sigma - \sigma_h\|_{L^2(\Omega)} \le Ch^{k+1}|\sigma|_{H^{k+1}(\Omega)}.$$

Remark 30. The extension to nearly incompressible or incompressible elastic materials is possible. In the homogeneous isotropic case the compliance tensor is given by

$$A\tau = \frac{1}{2\mu}\left(\tau - \frac{\lambda}{2\mu + n\lambda}\operatorname{tr}(\tau)\delta\right),$$

where δ is an identity matrix, and $\mu > 0$, $\lambda > 0$ are the Lamé constants. For our mixed method, as for most methods based on the Hellinger–Reissner principle, one can prove that the error estimates hold uniformly in λ. In the analysis above we use the fact that

$$\alpha\|\tau\|_0 \le (A\tau, \tau)$$

for some positive constant α. This estimate degenerates $\alpha \to 0$ when $\lambda \to +\infty$. However the estimate remains true with $\alpha > 0$ depending only on Ω and μ if we restrict τ to functions for which $\operatorname{div}\tau = 0$ and $\int_\Omega \operatorname{tr}(\tau)d\mathbf{x} = 0$, see [10], also [**?**, **?**] for more details.