16

Extended Finite Element Methods

The interface problems which involve partial differential equations having discontinuous coefficients across certain interfaces are often encountered in fluid dynamics, electromagnetics, and materials science. Because of the low global regularity and the irregular geometry of the interface, the standard numerical methods which are efficient for smooth solutions usually lead to loss in accuracy across the interface.

The interface-fitted/resolved grids may overcome this difficulty to some extent, but if the problem is time-dependent, the domain need to be re-meshed at each time step, which introduces interpolation error between two meshes. Before we move on to the interface-unfitted method, let us first give a summary of how accurate one could achieve numerically in an interface-fitted mesh for elliptic interface problems.

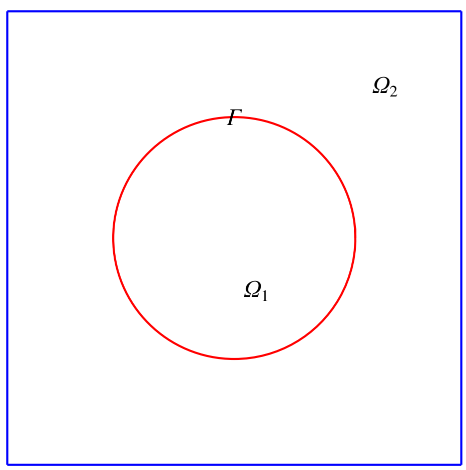

Fig. 16.1. Domain for an interface problem

To be specific, we consider the homogeneous boundary value problems of the diffusion equation

$$
\begin{cases}
-\nabla\cdot(a(x)\nabla u) = f, & \text{in } \Omega_1\cup\Omega_2,\\
[u]=0, \qquad [\alpha(x)\nabla u\cdot\mathbf{n}]=0, & \text{on } \Gamma,\\
u=0, & \text{on } \partial\Omega,
\end{cases}
\tag{16.1}
$$

where $[v] = v|_{\Omega_1} - v|_{\Omega_2}$ denotes the jump of v across the interface Γ, $\mathbf{n}$ is the unit outward normal to the boundary of Ω_1, and $\alpha(x)$ is bounded from below and above by some positive constants. Note that $\alpha(x)$ is allowed to be discontinuous across the interface Γ, which is assumed to be sufficiently smooth. To focus our attention on the curved interface Γ, we assume that Ω is polyhedral(polygonal). Fig. 16.1 is an illustration of a unit cube that contains a circle as an interface.

16.1 Interface-fitted method

A way to remedy the accuracy of approximation is to use interface-fitted/resolved grids. This way, the non-smoothness of the solution can be restricted to a "narrow" subdomain with respect to the grid near the interface, and the approximation error due to the non-smoothness can thus be dominated.

It is rather trivial to fit the interface when $d = 1$. In this case, the following error estimate can be easily obtained:

$$\|u - u_h|_{0,\Omega} + h|u - u_h|_{1,\Omega} \lesssim h^2|u|_{2,\Omega_1\cup\Omega_2}.$$

For $d \geq 2$, since Γ is curved in general, we can not have exact fitting. As it turns out, sharp error estimates for this type of body-fitted meshes is not as obvious as one might think.

In Xu (1982) [43] (see also [44] for an English translation) and more recently in Chen and Zou (1998) [13], the following error estimate was obtained for $d = 2$:

$$\|u - u_h\|_{0,\Omega} + h|u - u_h|_{1,\Omega} \lesssim |\log h|^{1/2}h^2|u|_{2,\Omega_1\cup\Omega_2}.$$

The proofs in both [43] and [13] made use of the following refined Sobolev inequality for $d = 2$ (that was obtained in [43]):

$$\|v\|_{0,p,\Omega} \lesssim p^{1/2}\|v\|_{1,\Omega}, \forall p \in [1,\infty), v \in H^1(\Omega).$$

A sharper analysis was given in Bramble and King (1996)[6], also for $d = 2$, and they obtained the following sharp estimate without the logarithm factor:

$$\|u - u_h\|_{0,\Omega} + h|u - u_h|_{1,\Omega} \lesssim h^2|u|_{2,\Omega_1\cup\Omega_2}.$$

The interface-fitting assumption in the works above can be loosened slightly to that the interface Γ is "$O(h^2)$-resolved by the mesh", see [28], and the shape-regularity restriction of the grid can be loosened to maximal-angle-bounded grids, see [12]. The optimal approximation accuracy of linear element space can also be proved on these grids.

16.2 XFEM: an overview

In an interface-fitted mesh, the sides (2D) or the edges (3D) intersect with the interface only through the vertices, see Figure 16.3. Unfortunately, it is usually a nontrivial and time-consuming task to construct good interface-fitted meshes for problems involving geometrically complicated interfaces. Therefore, numerous modified finite difference methods based only on simple Cartesian grids have been proposed in the literature. In the finite difference setting, we refer to the immersed boundary method in [36], the immersed interface method in [27, 29], the ghost fluid method in [31], and the references therein. In the finite element framework, we refer to the work of [30, 14] for elliptic problems with discontinuous coefficients in which finite element basis functions are locally modified for elements intersection the interface where the coefficient jumps.

In the past decade, a combination of unfitted finite elements (or XFEM) with the Nitsche method has become a popular discretization method for this type of interface problems. The extended finite element method (XFEM) was developed by Belytschko and his collaborators [3, 19] in 1999, to represent the crack independently of the mesh. We refer to [21] and the reference therein for a historical account for XFEM. Inspired by the simple idea of [34] for handling Dirichlet boundary conditions, Hansbo[23] applied Nitsche's method to reformulate an elliptic interface problem. It was proved that this Nitsche-XFEM achieved optimal convergence order for linear element. Nitsche's method is a variation of the Discontinuous Galerkin method applied at the interface, and is currently undergoing a revival, partly because it allows more freedom in the choice of approximation than the standard conforming finite element method.

In this short note, let us take elliptic interface problem (16.1) as an example to show how Nitsche-XFEM works [23].

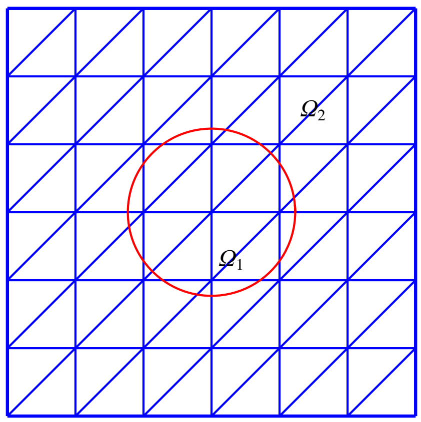

Fig. 16.2. Uniform, not interface-fitted

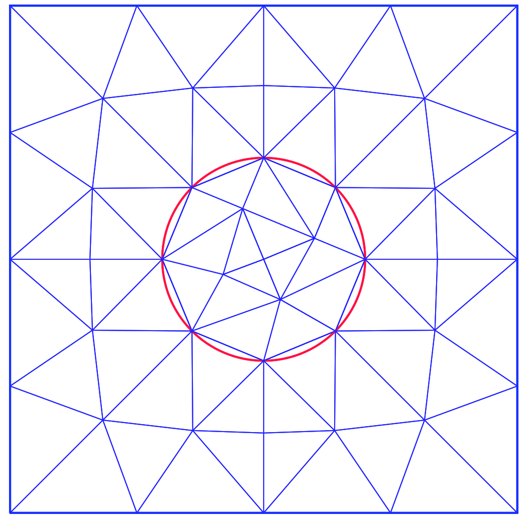

Fig. 16.3. Interface-fitted, shape regular.

16.3 Notation and XFEM formulation

Nitsche-XFEM method solves (16.1) approximately using piecewise polynomials on a family of conforming triangulations $\mathcal{T}_h$ of Ω which are independent of the location of the interface Γ while allows the approximation to be discontinuous inside elements which intersect the interface.

We will use the following notation for mesh related quantities. Let $\mathcal{T}_h^\Gamma = \{K \in \mathcal{T}_h, K \cap \Gamma \neq \emptyset\}$ be the set of all triangles that are intersected by Γ. Let $e = K \cap \Gamma$ and $K_i = K \cap \Omega_i$ with $i = 1, 2$. We define the average $\{\cdot\}$ on e by

$$\{v\} = \kappa_1 v_1 + \kappa_2 v_2$$

where $\kappa_i = \frac{|K_i|}{|K|}$ is the weight as suggested in [23], and $\kappa_1 + \kappa_2 = 1$.
A few remarks:

(i) Indeed, one may use any convex combination $\kappa v_1 + (1-\kappa)v_2, 0 \leq \kappa \leq 1$ in the definition of the average $\{\cdot\}$, without upsetting the consistency of the method.
(ii) To guarantee stability of this method, further conditions on the combinations of numerical fluxes must be imposed by choosing appropriate mesh and geometry dependent weights κ. One simple approach is to choose the numerical fluxes by

$$\kappa_1 = \begin{cases} 1, & \text{if } |K_1| \geq |K_2|, \\ 0, & \text{if } |K_1| < |K_2|. \end{cases}, \qquad \text{and} \quad \kappa_2 = 1 - \kappa_1, \tag{16.2}$$

Thus, for an intersected element, we compute the numerical quantity at that side of the interface where the larger part of the element resides. One can easily show the consistency and stability of the method with this selection of weights.

Let χ_i be the characteristic function on Ω_i with $i = 1, 2$. Given a mesh $\mathcal{T}_h$, let V_h be the continuous piecewise polynomial function space of degree $p \geq 1$ on the mesh. Let $V_h^1 := V_h \cdot \chi_1$ and $V_h^2 := V_h \cdot \chi_2$. Define the XFEM space by $V_h^\Gamma = V_h^1 + V_h^2$. In this way, we replace each standard basis function living on an element that intersects the interface by two new basis functions, namely its restrictions to Ω_1 and Ω_2, respectively. The XFEM scheme for interface problem is: Find $u_h \in V_h^\Gamma$, such that

$$a_h(u_h, v_h) = \int_\Omega f v_h\, dx, \quad \forall v_h \in V_h^\Gamma. \tag{16.3}$$

where

$$a_h(u,v) = \int_{\Omega_1\cup\Omega_2} \alpha(x)\nabla u \cdot \nabla v\, dx - \int_\Gamma \{\alpha(x)\nabla u \cdot \mathbf{n}\} \cdot [v]\, ds$$
$$- \int_\Gamma [u] \cdot \{\alpha(x)\nabla v \cdot \mathbf{n}\}\, ds + \frac{\gamma}{h}\int_\Gamma [u]\cdot[v]\, ds.$$

Here γ is a nonnegative number to be specified later.

16.4 Consistency

With these definitions, the discrete problem (16.3) is consistent in the sense that:

Lemma 117 (Consistency). *Let u be the exact solution of the interface problem* (16.1)*, then*

$$a_h(u, v_h) = \int_\Omega f\, v_h\, dx \quad \forall\, v_h \in V_h^\Gamma, \tag{16.4}$$

Proof. Note that $[u] = 0$ on Γ, by (16.1), we have

$$\begin{aligned} a_h(u, v_h) &:= \int_{\Omega_1\cup\Omega_2} \alpha(x)\nabla u \cdot \nabla v_h\, dx - \int_\Gamma \{\alpha(x)\nabla u\cdot\mathbf{n}\}\cdot[v_h]\, ds \\ &= \int_{\Omega_1\cup\Omega_2} -\nabla\cdot(\alpha(x)\nabla u)v_h\, dx + \int_\Gamma [\alpha(x)\nabla u\cdot \mathbf{n}v_h]\, ds - \int_\Gamma \{\alpha(x)\nabla u\cdot\mathbf{n}\}\cdot[v_h]\, ds \\ &= \int_{\Omega_1\cup\Omega_2} -\nabla\cdot(\alpha(x)\nabla u)v_h\, dx + \int_\Gamma [\alpha(x)\nabla u\cdot\mathbf{n}]\{v_h\}\, ds \\ &= \int_\Omega f\, v_h\, dx, \end{aligned}$$

For the third identity, we have used the fact that $[vw] = [v]\cdot\{w\} + \{v\}[w] + (\kappa_2 - \kappa_1)[v][w]$. This completes the proof. □

With the consistency (16.4), it is easy to see that the Galerkin orthogonality holds

$$a_h(u - u_h, v_h) = 0 \quad \forall\, v_h \in V_h^\Gamma, \tag{16.5}$$

To consider the boundedness and stability of the bilinear form $a_h(\cdot,\cdot)$, let $V(h) = V_h^\Gamma + H_0^1(\Omega_1 \cap \Omega_2)$, and define

$$|v|^2_{1,\Omega_1\cup\Omega_2} = \|\alpha(x)^{1/2}\nabla v\|^2_{0,\Omega_1\cup\Omega_2}, \quad |v|^2_{1,*} = \sum_{K\in\mathcal{T}_h^\Gamma} \gamma h_K^{-1}\|[v]\|^2_{0,e},$$

$$\|v\|_*^2 = |v|^2_{1,\Omega_1\cup\Omega_2} + \sum_{K\in\mathcal{T}_h^\Gamma} h_K\|\{\alpha(x)\nabla v\cdot\mathbf{n}\}\|^2_{0,e} + |v|^2_{1,*}. \tag{16.6}$$

Here, $e = K\cap\Gamma$.

Notice that the norm $\|v\|_*$ is good to obtain boundedness of the bilinear form $a_h(\cdot,\cdot)$ while the weaker norm $\|v\|_w = (|v|^2_{1,\Omega_1\cup\Omega_2} + |v|^2_{1,*})^{1/2}$ is the natural choice for analyzing the stability of XFEM methods. These two norms are equivalent on V_h^Γ with appropriate selection of penalty γ, actually, $\sum_{K\in\mathcal{T}_h^\Gamma} h_K\|\{\nabla v\}\|^2_{0,e}$ can be bounded by $\|v\|_w^2$.

We now have to show that the approximation property of V_h^Γ is still optimal in this mesh dependent norm.

16.5 Approximation ability of V_h^Γ

We wish to show that functions in V_h^Γ approximates functions $v \in H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)$ to the order h in the norm $\|v\|_*$. For this purpose, we construct an interpolant of v by nodal interpolants of H^2-extensions of v_1 and v_2 as follows. Let $s \geq 2$ be an integer and choose extension operators $E_1 : H^s(\Omega_1) \mapsto H^s(\Omega) \cap H_0^1(\Omega)$ and $E_2 : \{w \in H^s(\Omega_2) : \ w|_{\partial\Omega} = 0\} \mapsto H^s(\Omega) \cap H_0^1(\Omega)$ such that

$$(E_i w)|_{\Omega_i} = w \quad \text{and} \quad \|E_i w\|_{H^s(\Omega)} \lesssim \|w\|_{H^s(\Omega_i)}, \quad i = 1, 2.$$

Let I_h be the interpolation associated with V_h and denote $E_i w$ by $\tilde{w}_i$. Define an interpolation of u to V_h^Γ by

$$\Pi_h u = \chi_1 I_h \tilde{u}_1 + \chi_2 I_h \tilde{u}_2. \tag{16.7}$$

Let $\mu = \min\{p + 1, s\}$, then we have

$$\begin{aligned} \|u - \Pi_h u\|_*^2 &= |u - \Pi_h u|_{1,\Omega_1\cup\Omega_2}^2 + \sum_{K\in\mathcal{T}_h^\Gamma} h_K \|\{\alpha(x)\nabla(u - \Pi_h u)\cdot \mathbf{n}\}\|_{0,e}^2 + |u - \Pi_h u|_{1,*}^2 \\ &\leq C\gamma h^{2(\mu-1)} |u|_{s,\Omega_1\cup\Omega_2}^2. \end{aligned} \tag{16.8}$$

Here, to handle the edge terms at the interface, we need the following variant of the trace inequality: there exists a constant C, depending on Γ but independent of the mesh, such that

$$\|\nabla w\|_{0,K\cap\Gamma}^2 \leq C(h_K^{-1}|w|_{1,K}^2 + h_K |w|_{2,K}^2). \tag{16.9}$$

The crucial fact is that the constant in this inequality is independent of the location of the interface relative to the mesh. By applying the above trace inequality, we have

$$\begin{aligned} h_K \|\{\alpha(x)\nabla(u - \Pi_h u)\cdot \mathbf{n}\}\|_{0,e}^2 &\leq \sum_{i=1,2} h_K \|\chi_i \nabla(u - I_h u)\|_{0,e}^2 = \sum_{i=1,2} h_K \|\nabla(\tilde{u}_i - I_h \tilde{u}_i)\|_{0,e}^2 \\ &\leq \sum_{i=1,2} \left(\|\tilde{u}_i - I_h\tilde{u}_i\|_{1,K_i}^2 + h_K^2 \|\tilde{u}_i - I_h \tilde{u}_i\|_{2,K_i}^2 \right) \end{aligned}$$

and

$$\begin{aligned} h_K^{-1} \|[u - \Pi_h u]\|_{0,e}^2 &\leq \sum_{i=1,2} h_K^{-1} \|\chi_i (u - I_h u)\|_{0,e}^2 = \sum_{i=1,2} h_K^{-1} \|\tilde{u}_i - I_h \tilde{u}_i\|_{0,e}^2 \\ &\leq \sum_{i=1,2} \left(\|\tilde{u}_i - I_h\tilde{u}_i\|_{0,K_i}^2 + h_K^2 \|\tilde{u}_i - I_h \tilde{u}_i\|_{1,K_i}^2 \right) \end{aligned}$$

The estimates of the edge terms are thus reduced to those of bulk terms, which follow the standard interpolation arguments. We refer to [23, 42] for details of the proof. In fact, we can modify (16.9) in a way by replacing K in the right hand side with its larger sub-element K_i with $i = 1$ or 2. Thus the alternative definition of κ_i as in (16.2) is feasible, see [42] for many other choices. Optimal interpolation estimates follow, as does optimal convergence of the method irrespective of the location of the interface relative to the mesh.

16.6 Boundedness and stability

The following two lemmas give the continuity and coercivity of the bilinear form $a_h(\cdot, \cdot)$ of (16.3).

Lemma 118 (Boundedness).

$$a_h(u,v) \le 2\|u\|_* \, \|v\|_*, \quad \forall\, u,v \in V(h), \tag{16.10}$$

Proof. That is a direct consequence of the definitions (16.6) and the Cauchy-Schwarz inequality. □
For the stability, we have the following results.

Lemma 119 (Stability). *For linear finite elements, we have*

$$a_h(v,v) \ge C_s \|v\|_*^2, \quad \forall\, v \in V_h^{\Gamma}, \tag{16.11}$$

provided γ is chosen sufficiently large, where C_s is a positive constant depending on the angle condition of the mesh $\mathcal{T}_h$ but independent of the location of the interface relative to the mesh.

Proof. By the Cauchy-Schwarz inequality, we know that

$$\int_{\Gamma\cap K} \{\alpha(x)\nabla u\cdot \mathbf{n}\}\cdot [v]\, ds \le h_K^{\frac{1}{2}} \|\{\alpha(x)\nabla u\cdot\mathbf{n}\}\|_{0,\Gamma\cap K} \cdot h_K^{-\frac{1}{2}} \|[v]\|_{0,\Gamma\cap K}$$

Denote $\Gamma\cap K$ by Γ_K, then

$$\begin{aligned}
a_h(v,v) &= \|v\|_*^2 - \sum_{K\in\mathcal{T}_h^{\Gamma}} h_K \|\{\alpha(x)\nabla v\cdot\mathbf{n}\}\|_{0,\Gamma_K}^2 - 2\int_{\Gamma\cap K} \{\alpha(x)\nabla u\cdot\mathbf{n}\}\cdot[v]\, ds \\
&\ge \|\alpha(x)^{1/2}\nabla v\|_{0,\Omega_1\cup\Omega_2}^2 + |v|_{1,*}^2 - 2\sum_{K\in\mathcal{T}_h^{\Gamma}} h_K^{\frac{1}{2}} \|\{\alpha(x)\nabla v\cdot\mathbf{n}\}\|_{0,\Gamma_K}\cdot h_K^{-\frac{1}{2}}\|[v]\|_{0,\Gamma_K} \\
&\ge \|\alpha(x)^{1/2}\nabla v\|_{0,\Omega_1\cup\Omega_2}^2 + |v|_{1,*}^2 - 2(\epsilon\sum_{K\in\mathcal{T}_h^{\Gamma}} h_K\|\{\alpha(x)\nabla v\cdot\mathbf{n}\}\|_{0,\Gamma_K}^2 + \frac{|v|_{1,*}^2}{4\epsilon\gamma}) \\
&= \|\alpha(x)^{1/2}\nabla v\|_{0,\Omega_1\cup\Omega_2}^2 + (1-\frac{1}{2\epsilon\gamma})|v|_{1,*}^2 - 2\epsilon\sum_{K\in\mathcal{T}_h^{\Gamma}} h_K\|\{\alpha(x)\nabla v\cdot\mathbf{n}\}\|_{0,\Gamma_K}^2,
\end{aligned}$$

where $0<\epsilon\lesssim 1$ is an arbitrary constant number. Since $\nabla v\cdot\mathbf{n}$ is constant on K_i, we have

$$\begin{aligned}
h_K\|\kappa_i\nabla v_i\cdot\mathbf{n}\|_{0,\Gamma_K}^2 &\le h_K\kappa_i^2|\Gamma_K||\nabla v_i|^2 = h_K\kappa_i^2\frac{|\Gamma_K|}{|K_i|}\|\nabla v_i\|_{0,K_i}^2 \\
&= h_K\frac{|\Gamma_K||K_i|}{|K|^2}\|\nabla v_i\|_{0,K_i}^2 \le \tilde{C}\|\nabla v_i\|_{0,K_i}^2.
\end{aligned}$$

Then we have

$$\sum_{K\in\mathcal{T}_h^{\Gamma}} h_K\|\{\alpha(x)\nabla v\cdot\mathbf{n}\}\|_{0,\Gamma_K}^2 \le \tilde{C}\frac{\max\alpha(x)^2}{\min\alpha(x)}\|\alpha(x)^{1/2}\nabla v_i\|_{0,\Omega_1\cup\Omega_2}^2.$$

If ϵ is small enough, say, $\epsilon\simeq\frac{\min_{x\in\Omega}\alpha(x)}{4\tilde{C}\max_{x\in\Omega}\alpha(x)^2}$, we have

$$a_h(v,v) \ge \frac{1}{2}(|v|_{1,h}^2 + |v|_{1,*}^2) \ge C_s\|v\|_*^2. \tag{16.12}$$

if we choose $\gamma > \frac{1}{\epsilon}$. In addition, we could have $C_s \gtrsim \frac{1}{2(1+4\epsilon)}$. □

Notice that (16.11) only claims the coercivity of the bilinear form a_h on V_h.

16.7 Error Analysis for XFEM methods

Theorem 118. *Let u and u_h be the solutions of* (16.1) *and* (16.3) *respectively. Assume $u \in H^2(\Omega_1) \cup H^2(\Omega_2)$, the following a priori error estimates hold*

$$\|u - u_h\|_* \leq C\gamma^{\frac{1}{2}} h|u|_{2,\Omega_1 \cup \Omega_2}, \tag{16.13}$$

where C is a positive constant depending on the angle condition of the mesh $\mathcal{T}_h$ but independent of the location of the interface relative to the mesh.

Proof. Let $\Pi_h u \in V_h^\Gamma$ be the linear interpolant of u defined in (16.7). Recall the boundedness and stability of the bilinear form $a_h(\cdot,\cdot)$. We have

$$C_s\|\Pi_h u - u_h\|_*^2 \leq a_h(\Pi_h u - u_h, \Pi_h u - u_h) = a_h(\Pi_h u - u, \Pi_h u - u_h), \tag{16.14}$$

where we use the Galerkin orthogonality (16.5) to derive the last identity.

The proof follows from the triangle inequality and (16.8)

$$\|u - u_h\|_* \leq \|u - \Pi_h u\|_* + \|\Pi_h u - u_h\|_* \leq (1 + 2/C_s)\|u - \Pi_h u\|_* \leq C\gamma^{\frac{1}{2}} h|u|_{2,\Omega_1\cup\Omega_2}. \tag{16.15}$$

□

The same technique could be extended to high order finite elements. We refer to [42] for an hp version of XFEM scheme, of which the error estimates are optimal with respect to h and suboptimal with respect to p by half an order of p.