8

The sparse grid method

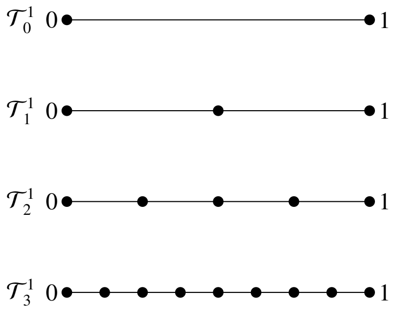
It is well-known that a smooth function defined on $[0,1]^d$ can be pointwisely approximated with $O(h^2)$ accuracy by a piecewise bilinear function in a subspace V_h of dimension $O(h^{-d})$. In the socalled sparse grid method proposed by Zenger, an $O(h^2|\log h|^{d-1})$ pointwise accuracy can be achieved by using a substantially smaller subspace $S_h \subset V_h$ of dimension $O(h^{-1}|\log h|^{d-1})$. As a result, a function u on a general domain in $\mathbb{R}^d$ can be approximated with $O(h|\log h|^{d-1})$ in H^1 norm with only $O(h^{-1}|\log h|^{d-1})$ number of operations.

8.1 Multi-linear elements

For simplicity, we take

$$D_s = (0,1)^s, \ 1 \le s \le d.$$

For D_1, let the first grid $\mathcal{T}_0^1$ be itself. Divide each element of $\mathcal{T}_k^1$ into two equal intervals and obtain the next level grid $\mathcal{T}_{i+1}^1$. Consider a cubic grid $\mathcal{T}_k^s$ of the domain D_s, which is a tensor product of $\mathcal{T}_k^1$. The vertice of elements there are $(i_1h_k, \cdots, i_sh_k)$, with $1 \le i_1, \cdots, i_s \le n_k - 1$, $n_k = h_k^{-1}$ and $h_k = 2^k$. For $s = 1$, denote



the basis function of nodal value interpolation to the vertex $x_{k,i} = ih_k$ by $\phi_{k,i}(x)$ with $1 \le i \le n_k - 1$. For $2 \le s \le d$, let

$$\mathbf{i} = (i_1, \cdots, i_s) \text{ and } x_{k,\mathbf{i}} = (i_1h_k, \cdots, i_sh_k)$$

and $\psi_{k,\mathbf{i}}^s(x_1, \cdots, x_s)$ be the basis function of the nodal value interpolation(bilinear interpolation) corresponding to the vertex $(i_1h_k, \cdots, i_sh_k)$ on the grid $\mathcal{V}_k$. Then

$$\psi^s_{k,\mathbf{i}}(x)=\prod_{j=1}^{s}\phi^j_{k,i_j}(x_j),\quad 1\le i_j\le n_k-1,\quad 1\le s\le d.$$

Here we add a superscript j to each basis function ϕ_{k,i_j} in one-dimensional case to represent that the basis function is the function of x_j component of $x=(x_1,\cdots,x_s)$. In other words, the grid $\mathcal{T}^s_k$ is obtained by cutting each cubic of $\mathcal{T}^s_{k-1}$ into 2^s equal cubics. Then the number of elements in $\mathcal{T}^s_k$ is $(n_k)^s=2^{ks}$ and the number of interior nodes is $(n_k-1)^s=(2^k-1)^s$.

Lemma 58. *Assume that $\Pi^s_k:C(\bar{D}_s)\mapsto\mathcal{T}^s_k(D_s)$ is the nodal value interpolant on $\mathcal{T}_k$ and I^s_k is the nodal value interpolant with respect to the variable x_s. Then*

$$\Pi^d_k u=\prod_{s=1}^{d}I^s_k.$$

Proof. For each $1\le s\le d$, let

$$\mathcal{I}^s_k=\{\mathbf{i}=(i_1,\cdots,i_s),\ 1\le i_1,\cdots,i_s\le n_k-1\}.$$

$$\begin{aligned}\Pi^d_k u&=\sum_{\mathbf{i}\in\mathcal{I}^d_k}u(x_{k,\mathbf{i}})\psi^d_{k,\mathbf{i}}\\&=\sum_{\mathbf{i}\in\mathcal{I}^d_k}u(x_{k,\mathbf{i}})\prod_{s=1}^{d}\phi^s_{k,i_s}(x_s)\\&=\sum_{\mathbf{i}\in\mathcal{I}^d_k}\prod_{s=1}^{d-1}\phi^s_{i_s}(x_s)\sum_{1\le i_d\le n_k-1}u(x_{k,\mathbf{i}})\phi^d_{i_d}(x_d)\qquad(8.1)\\&=\sum_{\mathbf{i}\in\mathcal{I}^{d-1}_k}\prod_{s=1}^{d-1}\phi^s_{i_s}(x_s)(I^d_k u)(x_{k,\mathbf{i}})\\&=(\prod_{s=1}^{d}I^s_k)u.\end{aligned}$$

□

Consider one-dimensional HB functions, we rewrite the nodal value interpolation operator I_k on $\mathcal{T}^1_J$ in 1-dimension as

$$I_J=\sum_{k=0}^{J}I_k-I_{k-1},$$

with $I_{-1}=0$. Let $\mathcal{M}=H^1_0(D_1)$ and

$$\mathcal{V}_k=(I_k-I_{k-1})\mathcal{M}=(I-I_{k-1})\mathcal{M}_k,\ k=0:J.\qquad(8.2)$$

It is easy to see that

$$\mathcal{V}_k=\{\phi_{k,i}:x_{k,i}\in\mathcal{N}_k\setminus\mathcal{N}_{k-1}\}.\qquad(8.3)$$

It is obvious that the dimension of $\mathcal{T}^1_k$ is 2^k. The above subspaces obviously give rise to a direct sum decomposition of the space $\mathcal{M}_J$ as follows:

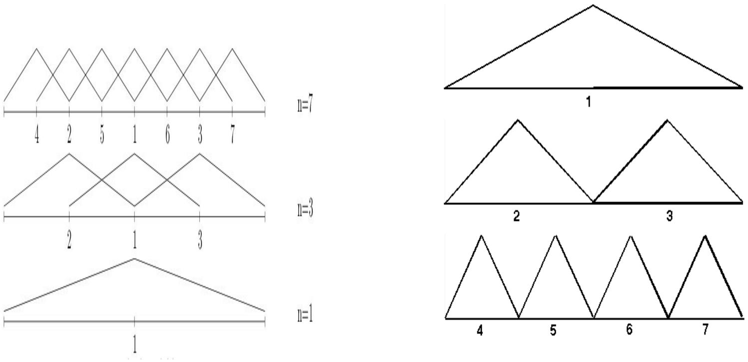


Fig. 8.1. Multilevel bases and hierarchical bases

$$\mathcal{M}_J = \oplus_{k=0}^{J} \mathcal{V}_k.$$

From this one dimensional hierarchical basis, a multi-dimensional basis on the d-dimensional unite cube D_d is obtain by a tensor product construction:

$$
\begin{aligned}
\mathcal{M}_J^d &= \overbrace{\mathcal{M}_J \otimes \mathcal{M}_J \otimes \cdots \otimes \mathcal{M}_J}^{d} \\
&= \oplus_{k_1,k_2,\cdots k_d=0}^{J} (\mathcal{V}_{k_1}^1 \otimes \mathcal{V}_{k_2}^2 \otimes \cdots \otimes \mathcal{V}_{k_d}^d) \\
&= \oplus_{j=0}^{dJ} \oplus_{k_1+k_2+\cdots+k_d=j} (\mathcal{V}_{k_1}^1 \otimes \mathcal{V}_{k_2}^2 \otimes \cdots \otimes \mathcal{V}_{k_d}^d)
\end{aligned}
\tag{8.4}
$$

Let $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ be a multi-index. Denote

$$\mathcal{V}_{\mathbf{k}} = \mathcal{V}_{k_1}^1 \otimes \mathcal{V}_{k_2}^2 \otimes \cdots \otimes \mathcal{V}_{k_d}^d.$$

Next we consider the approximation property and dimension of the following subspace:

$$S_{J,r}^d = \otimes_{|\mathbf{k}|_1 \le r} \mathcal{V}_{\mathbf{k}}$$

For the linear finite element, $\|u - u_I\|_0 \lesssim h^2 |u|_2$. Since $h \approx N^{-1/d}$, where N is the number of the total degree of freedom, we have

$$\|u - u_I\|_0 \lesssim N^{-2/d} |u|_2.$$

It implies that for a fixed number of degrees of freedom N, a large dimension results in pretty low accuracy.

In order to overcome the curse of dimensionality, we intend to choose a small subspace properly and throw away a large number of useless information. In the following, we introduce an interpolation operator R_h which interpolates functions into finite element spaces on coarse meshes. Denote the range of R_k^d by S_k^d, which is the desirable subspace.

Denote

$$
\begin{aligned}
\mathcal{F}_{J,r} &= \{\mathbf{k} = (k_1, \cdots, k_d) : 0 \le k_i \le J,\ |\mathbf{k}|_1 \le r\}, \\
\mathcal{F}_{J,r}^c &= \{\mathbf{k} = (k_1, \cdots, k_d) : 0 \le k_i \le J,\ |\mathbf{k}|_1 > r\}, \\
\mathcal{F}_{J,r}^o &= \{\mathbf{k} = (k_1, \cdots, k_d) : 0 \le k_i \le J,\ |\mathbf{k}|_1 = r\}.
\end{aligned}
$$

Lemma 59. *Assume that $I_k^s : C(\bar{D}) \mapsto \mathcal{M}_l(D_1)$ is the nodal value interpolant with respect to the variable x_s. Define*

$$F_{J,r}^d = \sum_{\mathbf{k}\in\mathcal{F}_{J,r}} \prod_{s=1}^d (I_{k_s}^s - I_{k_{s-1}}^s)$$

and

$$S_{J,r}^d = F_{J,r}^d \mathcal{M}.$$

Then for $m = 0, 1$,

$$\inf_{\chi\in S_{J,r}^d} \|v-\chi\|_{m,D_d} \le C h_J^{(2-m)(r+1)/J} |\log h_J|^{d-1} \max_{\mathbf{k}\in\mathcal{F}_{J,r}^c} \|\partial^{2sign(\mathbf{k})} v\|_{0,D_d}.$$

Proof. Using the obvious identity (with $I_{-1}^s = 0$)

$$I_J^s = \sum_{k=0}^J (I_k^s - I_{k-1}^s),$$

it follows from the commutative property that

$$\Pi_J^d = \prod_{s=1}^d \sum_{k=0}^J (I_k^s - I_{k-1}^s) = \sum_{k_1,k_2,\cdots,k_d=0}^J \prod_{s=1}^d (I_{k_s}^s - I_{k_s-1}^s).$$

Thus

$$(\Pi_k^d - F_k^d)v = \sum_{\mathbf{k}\in\mathcal{F}_{J,r}^c} \prod_{s=1}^d (I_{k_s}^s - I_{k_s-1}^s)v. \tag{8.5}$$

Note that if $k_1 \neq 0$,

$$\begin{aligned} \|\prod_{s=1}^d (I_{k_s}^s - I_{k_s-1}^s)v\|_{m,D_d} &\lesssim h_{k_1}^{2-m} \|\partial_{x_1}^2 \prod_{s=2}^d (I_{k_s}^s - I_{k_s-1}^s)v\|_{0,D_d} \\ &= h_{k_1}^{2-m} \|\prod_{s=2}^d (I_{k_s}^s - I_{k_s-1}^s)\partial_{x_1}^2 v\|_{0,D_d} \\ &= 2^{-k_1(2-m)} \|\prod_{s=2}^d (I_{k_s}^s - I_{k_s-1}^s)\partial_{x_1}^2 v\|_{0,D_d}. \end{aligned} \tag{8.6}$$

If $k_1 = 0$,

$$\begin{aligned} \|\prod_{s=1}^d (I_{k_s}^s - I_{k_s-1}^s)v\|_{m,D_d} &= \|I_0^1 \prod_{s=2}^d (I_{k_s}^s - I_{k_s-1}^s)v\|_{m,D_d} \\ &\le \|\prod_{s=2}^d (I_{k_s}^s - I_{k_s-1}^s)v\|_{m,D_d} \\ &= 2^{-k_1(2-m)} \|\prod_{s=2}^d (I_{k_s}^s - I_{k_s-1}^s)v\|_{m,D_d}. \end{aligned} \tag{8.7}$$

A combination of (8.6) and (8.7) gives

$$\|\prod_{s=1}^{d}(I^s_{k_s} - I^s_{k_s-1})v\|_{0,D_d} \lesssim 2^{-(2-m)|\mathbf{k}|_1}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}. \tag{8.8}$$

An elementary calculation shows that

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{F}^c_{J,r}} 2^{-(2-m)|\mathbf{k}|_1}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d} &= \sum_{j=r+1}^{Jd}\sum_{\mathbf{k}\in\mathcal{F}^o_{J,j}} 2^{-(2-m)|\mathbf{k}|_1}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}\\ &\le \sum_{j=r+1}^{Jd} 2^{-(2-m)j}\max_{\mathbf{k}\in\mathcal{F}^o_{J,j}}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}\sum_{\mathbf{k}\in\mathcal{F}^o_{J,j}} 1\\ &\le J^{d-1}\sum_{j=r+1}^{Jd} 2^{-(2-m)j}\max_{\mathbf{k}\in\mathcal{F}^c_{J,r}}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}\\ &= \frac{1-2^{-(2-m)(Jd-r-1)}}{1-2^{-(2-m)}}J^{d-1}2^{-(r+1)(2-m)}\max_{\mathbf{k}\in\mathcal{F}^c_{J,r}}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}\\ &\le Ch_J^{(2-m)(r+1)/J}|\log h_J|^{d-1}\max_{\mathbf{k}\in\mathcal{F}^c_{J,r}}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}.\end{aligned} \tag{8.9}$$

A combination of (8.5), (8.8) and (8.9) yields

$$\|v - F^d_k v\|_{m,D_d} \le \|v - \Pi^d_k v\|_{m,D_d} + \|\Pi^d_k v - F^d_k v\|_{m,D_d} \le Ch_J^{(2-m)(r+1)/J}|\log h_J|^{d-1}\max_{\mathbf{k}\in\mathcal{F}^c_{J,r}}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d}.$$

which completes the proof. □

Remark 14. Let $\lceil p/2\rceil$ be the smallest integer larger than $p/2$. For any positive integer p, if $r \ge \lceil p/2\rceil J$, there are at least $\lceil p/2\rceil$ components of $\mathbf{k}$ which are nonzero, namely, $2\mathrm{sign}(\mathbf{k}) > p$. Thus, for any polynomial v with degree not larger than p, $\max_{\mathbf{k}\in\mathcal{F}^c_{J,r}}\|\partial^{2\mathrm{sign}(\mathbf{k})}v\|_{0,D_d} = 0$.

Lemma 60. *By the definition of $S^d_{J,r}$, we have*

$$S^d_{J,r} = \mathrm{span}\{\prod_{s=1}^{d}\phi^s_{k_s,i_s}, \quad |\mathbf{k}|_1 < r, \quad i_s h_{k_s} \in N_{k_s}\setminus N_{k_s-1}\}$$

and

$$\dim(S^d_{J,r}) = O(h_J^{-r/J}|\log h_J|^{d-1}).$$

Proof. It is easy to see that

$$\begin{aligned}\dim(S^d_{J,r}) &\le C\sum_{|\mathbf{k}|\le r} 2^{k_1+k_2+\cdots+k_d}\\ &= C\sum_{j=0}^{r} 2^j\#\mathcal{F}^o_{J,j}\\ &\le 2^r J^{d-1}\\ &\le Ch_J^{-r/J}|\log h_J|^{d-1}.\end{aligned}$$

□

8.2 The sparse grid method using high order polynomials

In this subsection, we consider the high order method. Let $\mathcal{M}_k, k = 0, 1, \cdots$ denote the one-dimensional Lagrange finite element spaces of order p on the grids $\mathcal{T}_k$. Note that the number of elements on $\mathcal{T}_k, k = 0, 1, \cdots$ is 2^0. Thus, the number of degrees of freedom of the space $\mathcal{M}_k$ is

$$\mathrm{Dim}(\mathcal{M}_k) = p2^k + 1 \quad \text{for } k \geq 1,$$

and $\mathrm{Dim}(\mathcal{M}_0) = p + 1$.

Setting $\mathcal{V}_0 = \mathcal{M}_0$ and

$$\mathcal{V}_k = (I_k - I_{k-1})\mathcal{M}_k \quad \text{for } k \geq 1.$$

Thus, the dimension of the spaces is

$$\mathrm{Dim}(\mathcal{V}_k) = p2^{k-1} \quad \text{for } k \geq 1,$$

and $\mathrm{Dim}(\mathcal{V}_0) = p + 1$. It is easy to see that

$$\mathcal{V}_k = \{\phi_{k,i} : x_{k,i} \in \mathcal{N}_k \setminus \mathcal{N}_{k-1}\}, \tag{8.10}$$

where $\mathcal{N}_k$ denotes the set of nodes of Lagrange finite element space of order k. The above subspaces obviously give rise to a direct sum decomposition of the space $\mathcal{M}_J$ as follows:

$$\mathcal{M}_\infty = \oplus_{k=0}^{\infty}\mathcal{V}_k.$$

From this one-dimensional basis, a multi-dimensional basis on the d-dimensional unite cube D_d is obtain by a tensor product construction:

$$\begin{aligned} \mathcal{M}^d &= \overbrace{\mathcal{M} \otimes \mathcal{M} \otimes \cdots \otimes \mathcal{M}}^{d} \\ &= \oplus_{k_1,k_2,\cdots k_d=0}^{\infty}(\mathcal{V}_{k_1}^1 \otimes \mathcal{V}_{k_2}^2 \otimes \cdots \otimes \mathcal{V}_{k_d}^d) \\ &= \oplus_{j=0}^{\infty} \oplus_{k_1+k_2+\cdots+k_d=j} (\mathcal{V}_{k_1}^1 \otimes \mathcal{V}_{k_2}^2 \otimes \cdots \otimes \mathcal{V}_{k_d}^d) \end{aligned} \tag{8.11}$$

Let $\mathbf{k} = (k_1, k_2, \cdots, k_d)$ be a multi-index. Denote $\mathcal{V}_{\mathbf{k}} = \mathcal{V}_{k_1}^1 \otimes \mathcal{V}_{k_2}^2 \otimes \cdots \otimes \mathcal{V}_{k_d}^d$. We make the following truncation:

$$\mathcal{M}^d = \left(\oplus_{|\mathbf{k}|_1 < L}\mathcal{V}_{\mathbf{k}}\right) \oplus \left(\oplus_{|\mathbf{k}|_1 \geq L}\mathcal{V}_{\mathbf{k}}\right).$$

Next we consider the approximation property and dimension of $\mathcal{M}^{d,L,p} := \oplus_{|\mathbf{k}|_1 < L}\mathcal{V}_{\mathbf{k}}$.

Lemma 61. *It holds that*

$$Dim(\mathcal{M}^{d,L,p}) = \mathcal{O}\Big((p + 1)^d L^{-1}(L + 2)^d 2^{L-d}\Big). \tag{8.12}$$

Proof.

$$\begin{aligned}
\mathrm{Dim}(\mathcal{M}^{d,L,p}) &= \mathrm{Dim}(\oplus_{|\mathbf{k}|_1<L}\mathcal{V}^1_{k_1}\otimes\mathcal{V}^2_{k_2}\otimes\cdots\otimes\mathcal{V}^d_{k_d})\\
&= \sum_{i=0}^{d} C_d^i(p+1)^i \sum_{k_1+\cdots+k_{d-i}<L}^{k_j\geq 1} p^{d-i}2^{k_1+\cdots+k_{d-i}-(d-i)}\\
&= \sum_{i=0}^{d} C_d^i(p+1)^i \sum_{k_1+\cdots+k_{d-i}<L-(d-i)}^{k_j\geq 0} p^{d-i}2^{k_1+\cdots+k_{d-i}}\\
&\leq \sum_{i=0}^{d} C_d^i(p+1)^i p^{d-i}(L-d+i)^{d-i-1}2^{L-d+i}\\
&\leq (p+1)^d L^{-1}2^{L-d}\sum_{i=0}^{d} C_d^i L^{d-i}2^i\\
&\leq (p+1)^d L^{-1}(L+2)^d 2^{L-d}.
\end{aligned}\tag{8.13}$$

In the first inequality of above estimation, we use the fact that

$$\sum_{k_1+\cdots+k_d<L} 2^{k_1+\cdots+k_d} = \sum_{j=0}^{L-1}\sum_{k_1+\cdots+k_d=j} 2^j = \sum_{j=0}^{L-1} C_{j+d-1}^{d-1}2^j,$$

and

$$\frac{(L-1+d-1)!}{(d-1)!(L-1)!}2^{L-1} < \sum_{j=0}^{L-1} C_{j+d-1}^{d-1}2^j < \sum_{j=0}^{L-1} L^{d-1}2^j = L^{d-1}2^L.$$

Thus,

$$\sum_{k_1+\cdots+k_d<L} 2^{k_1+\cdots+k_d} \approx L^{d-1}2^L.$$

□

Lemma 62. *It holds that*

$$\|\Pi_{i=1}^d(I_{k_i}^i - I_{k_i-1}^i)v\|_0 \leq C^d 2^{-(p+1)|\mathbf{k}|_1}\|\partial^{(p+1)sign(\mathbf{k})}v\|_0.\tag{8.14}$$

Here C is a constant independent of $p, d, \mathbf{k}$.

Proof.

$$\begin{aligned}
\|\Pi_{i=1}^d(I_{k_i}^i - I_{k_i-1}^i)v\|_0 &\leq Ch_{k_i}^{p+1}\|\Pi_{i=2}^d(I_{k_i}^i - I_{k_i-1}^i)\partial^{(p+1)\mathrm{sign}(k_1)}v\|_0\\
&\leq C^d 2^{-(p+1)|\mathbf{k}|_1}\|\partial^{(p+1)\mathrm{sign}(\mathbf{k})}v\|_0.
\end{aligned}\tag{8.15}$$

□

For any v, we denote $v_{\mathbf{k}} = \Pi_{i=1}^d(I_{k_i}^i - I_{k_i-1}^i)v$. It is obvious that

$$v \approx \sum_{\mathbf{k}} v_{\mathbf{k}}.$$

The above lemma indicates that $v_{\mathbf{k}}$ decays as $C^d 2^{-(p+1)|\mathbf{k}|_1}$

Lemma 63. *Setting* $v_{\mathbf{k}} = \Pi_{i=1}^d (I_{k_i}^i - I_{k_i-1}^i)v$*. It hold that*

$$\|\sum_{|\mathbf{k}|_1 \geq L} v_{\mathbf{k}}\|_0 \leq C^d 2^{-(p+1)L + \frac{1}{2^{p+1}\ln 2}d} \sum_{i=0}^{d-1} \frac{\left((1 - \frac{1}{2^{p+1}})L\right)^i}{i!} \max_{|\mathbf{k}|_1 \geq L} \|\partial^{(p+1)sign(\mathbf{k})} v\|_0 \tag{8.16}$$

Proof. By Lemma 62, we have

$$\|\sum_{|\mathbf{k}|_1 \geq L} v_{\mathbf{k}}\|_0 \leq C^d \max_{|\mathbf{k}|_1 \geq L} \|\partial^{(p+1)\mathrm{sign}(\mathbf{k})} v\|_0 \sum_{\mathbf{k}|_1 \geq L} 2^{-(p+1)\mathbf{k}}.$$

Setting $s_{d,L} = \sum_{\mathbf{k}|_1 \geq L} 2^{-(p+1)\mathbf{k}}$. Note that

$$\begin{aligned} s_{d,L} &= \sum_{\mathbf{k}|_1 \geq L} 2^{-(p+1)\mathbf{k}} \\ &= \sum_{k_1 + \cdots + k_d \geq L-1} 2^{-(p+1)(k_1+1+k_2+\cdots+k_d)} + \sum_{0+k_2+\cdots+k_d \geq L} 2^{-(p+1)(0+k_2+\cdots+k_d)} \\ &= 2^{-(p+1)} s_{d,L-1} + s_{d-1,L} \end{aligned}$$

and $s_{1,L} = \frac{2^{-(p+1)L}}{1-2^{-p-1}}$, $s_{d,0} = (\frac{1}{1-2^{-p-1}})^d$.

Let $t_{d,L} = 2^{(p+1)L} s_{d,L}$, $\gamma = \frac{1}{1-2^{-p-1}}$. We have the following induction

$$t_{d,L} = t_{d,L-1} + t_{d-1,L} \quad \text{with} \quad s_{1,L} = \gamma, s_{d,0} = \gamma^d.$$

By induction, we have

$$t_{d,L} \lesssim \gamma^d (1 + \frac{L}{\gamma} + \cdots + \frac{L^{d-1}}{(d-1)!\gamma^{d-1}}).$$

Thus,

$$s_{d,L} \lesssim 2^{-(p+1)L}(1 - 2^{-p-1})^{-d}(1 + \frac{(1 - 2^{-p-1})L}{1} + \cdots + \frac{\left((1 - 2^{-p-1})L\right)^{d-1}}{(d-1)!})$$

Moreover, we have

$$(1 - 2^{-p-1})^{-d} = 2^{-d\ln(1-2^{-p-1})/\ln 2} \leq 2^{\frac{1}{2^{p+1}\ln 2}d}.$$

It ends the proof. □