5

H(grad), H(curl) and H(div) spaces

The Laplacian operator is the most important example of partial differential operator of elliptic type. In this chapter, we will present some main results on this operator.

5.1 Dual of $H_0^1(\Omega)$: $H^{-1}(\Omega)$

Using the definition of Sobolev space above, we define $H_0^1(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ with respect to the $\|\cdot\|_{1,\Omega}$ norm for H^1 function defined by:

$$\|v\|_{1,\Omega}^2 = \|v\|_{0,\Omega}^2 + \|\nabla v\|_{0,\Omega}^2.$$

$H_0^1(\Omega)$ is a subspace of $H^1(\Omega)$ and it can be characterized as

$$H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}.$$

On $H_0^1(\Omega)$, thanks to the Poincare inequality

$$\|v\|_{0,\Omega} \lesssim \|\nabla v\|_{0,\Omega}, \quad v \in H_0^1(\Omega), \tag{5.1}$$

the the semi-norm $|v|_{1,\Omega} = \|\nabla v\|_{0,\Omega}$ defines an equivalent norm and $(\nabla u, \nabla v)$ defines on equivalent inner product on $H_0^1(\Omega)$.

As a Hilbert space, $(H_0^1(\Omega))'$ has a natural representation, namely $H_0^1(\Omega)$ itself, with the H^1-inner product as paring. We now discuss that, when the L^2-paring is used, what the corresponding representation of $(H_0^1(\Omega))'$ is.

In fact, for any $f \in L^2(\Omega)$, the L^2-pairing

$$\langle f, v\rangle = (f, v)$$

indeed defines a continuous linear function on $H_0^1(\Omega)$, thanks to the Poincare inequality (5.10). Thus we can view f as an element in V' and define its corresponding dual norm

$$\|f\|_{-1,\Omega} = \sup_{v\in H_0^1(\Omega)} \frac{(f, v)}{|v|_{1,\Omega}}. \tag{5.2}$$

The $H^{-1}(\Omega)$ is then defined to be the closure of $L^2(\Omega)$ with respect to the above $\|\cdot\|_{-1,\Omega}$ norm. It is easy to see that $H^{-1}(\Omega)$ is a representation of $(H_0^1(\Omega))'$. Since this dual results from the L^2-pairing, it is customary to write

$$(H_0^1(\Omega))' = H^{-1}(\Omega).$$

Given $f \in V'$, by Riesz representation Theorem, there exists unique $u \in V$ such that

$$(\nabla u, \nabla v) = \langle f, v\rangle, \quad v \in V. \tag{5.3}$$

Using the definition of weak derivatives, we have

$$\langle -\Delta u, v\rangle = (\nabla u, \nabla v), \quad v \in \mathcal{D}(\Omega)$$

Thus, in $\mathcal{D}'(\Omega)$, we have

$$-\Delta u = f \tag{5.4}$$

which also holds in V' since $\mathcal{D}(\Omega)$ is dense in V.

In summary, we have the following theorem.

Theorem 36. *The following mapping is an isomorphism*

$$-\Delta : H_0^1(\Omega) \mapsto H^{-1}(\Omega).$$

This is the most basic result in the variational theory of partial differential equations. It also give certain characterization of $H^{-1}(\Omega)$.

Now taking any $w \in L^2(\Omega)$. By definition of weak derivatives, we have

$$\langle \partial_i w, v\rangle = -(w, \partial_i v) \quad v \in H_0^1(\Omega).$$

This immediately implies that

$$\partial_i w \in H^{-1}(\Omega), \quad \|\partial_i w\|_{-1,\Omega} \le \|w\|_{0,\Omega}$$

In summary, we have

Theorem 37.

$$H^{-1}(\Omega) = \left\{-\Delta u : u \in H_0^1(\Omega)\right\} = \left\{w_0 + \sum_{i=1}^{d} \partial_i w_i : w_0, w_i \in L^2(\Omega)\right\}.$$

This characterization of $H^{-1}(\Omega)$ partially explains the origin of the notation itself since $L^2(\Omega) = H^0(\Omega)$.

The following inequality is non-trivial:

$$\|v\|_{0,\Omega} \lesssim \|\nabla v\|_{-1,\Omega} + \|v\|_{-1,\Omega}.$$

5.2 Elliptic boundary value problems and regularity

Consider

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{5.5}$$

Here, Ω is a bounded domain in R^d. The bilinear form corresponding to the operator $-\Delta$ is defined by

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v. \tag{5.6}$$

This form is defined for all v and w in the Sobolev space $H_0^1(\Omega)$. Clearly, $u \in H_0^1(\Omega)$ is the solution of

$$a(u,v) = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega). \tag{5.7}$$

The H^1 solution of (5.7) is often called a weak solution of (5.5). This solution u can be proved to be a solution of (5.5) in a more classic sense if u is smooth enough. The theory for proving the smoothness of the weak solution is called *regularity* theory. This sort of theory is often not very straightforward. We shall only give a brief account of this theory. But this theory is very important for the theory of finite element approximation and convergence of multigrid methods.

To get a rough idea of regularity theory, let us study a property of Laplacian operator on the whole space by using Fourier transform. Given a distribution v defined on $\mathbb{R}^n$ such that $\Delta v \in L^2$, by the properties of Fourier transform, we have

$$\widehat{D^\alpha v}(\xi) = (i\xi)^\alpha \hat{v}(\xi) = -(i\xi)^\alpha |\xi|^{-2} \widehat{\Delta v}(\xi).$$

The function $(i\xi)^\alpha|\xi|^{-2}$ is bounded by 1 if $|\alpha| = 2$, hence

$$\|\widehat{D^\alpha v}\|_{0,\mathbb{R}^d} \le \|\widehat{\Delta v}\|_{0,\mathbb{R}^d}.$$

By Planchel identidy, we have

$$\|D^\alpha v\|_{0,\mathbb{R}^d} \le \|\Delta v\|_{0,\mathbb{R}^d} \quad \forall\ |\alpha| = 2.$$

The above inequality illustrates an important fact that if v is a function such that $\Delta v \in L^2$, then all its second order derivatives are also in L^2. If we think about it a little, this is a rather significant fact since Δv is a very special combination of the second order derivatives of v. It is not easy to see this property of Laplacian also holds for the generalized elliptic operator if the coefficient functions are sufficiently smooth.

This property of elliptic operator can be extended to bounded domains with smooth boundary, but such an extension is not trivial. The following theorem is well-known and it can be found in most of the text books on elliptic boundary value problems.

Theorem 38. *Let Ω be a smooth and bounded domain of $\mathbb{R}^n$. Then for each $f \in L^2(\Omega$, there exists a unique $u \in H^2(\Omega)$, the solution of* (5.5)*, that satisfies*

$$\|u\|_{2,\Omega} \le C\|f\|_{0,\Omega}$$

where C is a positive constant depending on Ω.

The above regularity theorem, however, does not hold on general Lipschitz domains. To see this, let us give a simple counter example. Given $\beta \in (0,1)$, consider the following nonconvex domain

$$\Omega = \{(r,\theta) : 0 < r < 1, 0 < \theta < \pi/\beta\}.$$

Let $v = r^\beta \sin(\beta\theta)$. Being the imaginary part of the complex analytic functin z^β, v is harmonic in Ω. Define $u = (1 - r^2)v$. A direct calculation shows that

$$-\Delta u = 4(1+\beta)v \text{ in } \Omega \text{ and } u|_{\partial\Omega} = 0.$$

Note that $4(1+\beta)v \in L^\infty(\Omega) \subset L^2(\Omega)$, but $u \notin H^2(\Omega)$.

Nevertheless, a slightly weaker result does hold for general Lipschitz domains.

Theorem 39. *Assume that Ω is a bounded Lipschitz domain. Then there exists a constant $\alpha \in (0,1]$ such that*

$$\|u\|_{1+\alpha} \le C\|F\|_{\alpha-1}, \tag{5.8}$$

for the solution u of (5.7)*, where C is a constant depending on the domain Ω.*
Furthermore $\alpha = 1$ if Ω is convex.

We shall now give a proof in the case that Ω is convex for which the Theorem holds for $\alpha = 1$. For the proof of the above theorem for more general case, we refer to Grisvard [22].

Lemma 26. *Let Ω be a convex, bounded domain of $\mathbb{R}^d$. Then for any $u \in H_0^1(\Omega) \cap H^2(\Omega)$*

$$\sum_{i,j=1}^{d} \int_\Omega |\partial_{ij}^2 u|^2 \le \int_\Omega |\varDelta u|^2.$$

Proof. We first establish the inequality for $u \in H_0^1(\Omega) \cap C^3(\bar{\Omega})$. It follows from the Green formula that

$$\int_\Omega |\varDelta u|^2 - \sum_{i,j=1}^{d} \int_\Omega |\partial_{ij}^2 u|^2 = \int_{\partial\Omega} (\varDelta u \partial_\nu u u - \frac{1}{2}\partial_\nu |\nabla u|^2). \tag{5.9}$$

We shall prove the result under a slightly stronger assumption that Ω is piecewise smooth. Given a smooth point $x_0 \in \partial\Omega$. Assume that, around x_0, Γ is given by the graph of the function $x_d = g(x_1, \cdots, x_{d-1}), |x_i - x_i^0| < \delta, 0 \le i \le n-1$ for some $\delta > 0$. Since both $\varDelta u$ and $|\nabla u|$ are invariant under both rotation and translation, we may assume that $x^0 = 0$ and $\nu == (0, \ldots, 0, 1)^T$.

Let $\Phi = x_d - g(x_1, \cdots, x_{d-1})$. Then

$$\nu = \nabla\Phi/|\nabla\Phi|, \quad \partial_i g = -\partial_i \Phi = \nu_i/\nu_d = 0, \quad 1 \le i \le d-1.$$

Since $u \in H_0^1(\Omega)$, we have $u(x_1, \cdot, x_{d-1}, g(x_1, \cdots, x_{d-1})) \equiv 0$. Differentiating this identity and using the chain rule, we deduce that, for $1 \le i \le d-1$

$$\partial_i u = 0$$

and, for $1 \le i, j \le d-1$

$$\partial_{ij}^2 u = \partial_{ij}^2 \Phi \partial_d u.$$

It follows that

$$\varDelta u = \mathrm{tr}(\partial_d u)$$

and

$$\frac{1}{2}\partial_\nu |\nabla u|^2 = \sum_{i=1}^{d} \partial_i u \sum_{j=1}^{d} \partial_{ij}^2 u \nu_j = \partial_n u \partial_{nn}^2 u.$$

Consequently,

$$\begin{aligned} \varDelta u \partial_\nu u - \frac{1}{2}\partial_\nu |\nabla u|^2 &= (\partial_n u)(\partial_{nn}^2 u + \mathrm{tr}(D^2\Phi)\partial_n u) - \partial_n u \partial_{nn}^2 u \\ &= (\partial_n u)^2 \mathrm{tr}(D^2\Phi) \ge 0. \end{aligned}$$

The desired estimate then follows. □

Remark 12. We have in fact proved the following identity

$$\int_\Omega |\Delta u|^2 - \sum_{i,j=1}^d \int_\Omega |\partial_{ij}^2 u|^2 = \int_{\partial\Omega} H_{\partial\Omega} |\nabla u|^2 \quad \forall\ u \in H_0^1(\Omega) \cap H^2(\Omega)$$

where $H_{\partial\Omega}$ is the mean curvature function for the boundary of Ω.

Lemma 27. *Let Ω be a convex, bounded domain of $\mathbb{R}^d$. Then for any $u \in H_0^1(\Omega) \cap H^2(\Omega)$*

$$\|u\|_{2,\Omega} \le C(\Omega)\|\Delta u\|_{0,\Omega}$$

Proof. By a standard energy argument, $|v|_{1,\Omega} \le \|\Delta u\|_{0,\Omega}$. By Poincare inequality, $\|u\|_{1,\Omega} \le C(\text{diam}\Omega)|v|_{1,\Omega}$. The desired estiamte then follows by combining the previous lemma. □

Now we are ready to prove the theorem.

Proof. We choose a sequence Ω_m, $m = 1, 2, \ldots$ of convex subsets of Ω with C^2 bondaries $\partial\Omega_m$ so that dist $(\partial\Omega_m, \partial\Omega)$ tends to zero as m tends to infinity. By the well-known regularity theorem for smooth domain (cf. Gilbarg and Trudinger), for each m, there exists $u_m \in H_0^1(\Omega) \cap H^2(\Omega)$ such that $-\Delta u_m = F$ in Ω_m. By Lemma 27, there exists a constant C such that $\|u_m\|_{2,\Omega_m} \le C$. This implies that $\tilde{u}_m$ is a bounded sequence in $H^1(\mathbb{R}^d)$ and $v_{m,ij} = \widetilde{\partial_i\partial_j u_m}$ are bounded sequences in $L^2(\mathbb{R}^d)$ for $1 \le i, j \le d$. Consequently there exit $V \in H^1(\mathbb{R}^d)$ and $V_{ij} \in L^2(\mathbb{R}^d)$ and a suitable increasing sequence of integers m_k (k=1,2,...) such that, as $k \to \infty$

$$\tilde{u}_{m_k} \to V \text{ weakly in } H^1(\mathbb{R}^d), \quad \tilde{v}_{m_k,i,j} \to V_{i,j} \text{ weakly in } L^2(\mathbb{R}^d).$$

Let $u = V|_\Omega$. It is easy to check that $u \in H_0^1(\Omega)$ and satifies $(\nabla u, \nabla\phi) = (F, \phi)$ for all $\phi \in H_0^1(\Omega)$. It can also be easily checked, by definition, that $\partial_i\partial_j u = V_{ij}|_\Omega \in L^2(\Omega)$. Thus $u \in H^2(\Omega)$. □

5.2.1 The self-adjoint property of $-\Delta$

Let us discuss about the following operator

$$A = -\Delta$$

5.2.2 As a bounded operator

In the previous sections, we have shown the following isomorphic property of the operator $-\Delta$:

$$-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega).$$

Namely,

$$A \equiv -\Delta : V \to V'$$

with $V = H_0^1(\Omega)$.

The corresponding energy inner product is simply the inner product of $H_0^1(\Omega)$:

$$[u, v] = \langle Au, v\rangle = (\nabla u, \nabla v), \quad u, v \in V.$$

It is interesting to consider the inverse of $-\Delta$ which gives the following isomorphism:

$$(-\Delta)^{-1} : H^{-1}(\Omega) \to H_0^1(\Omega).$$

Since $H_0^1(\Omega)$ is compactly imbedded into $H^{-1}(\Omega)$, we have the following compact operator:

$$(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega).$$

which is also self-adjoint with respect to the L^2 inner product. [1] This observation can be used to analyze the spectrum property of $-\Delta$.

5.2.3 As an unbounded operator

In the literature, $-\Delta$ is often studied as a densely defined self-adjoint unbounded operator. Despite that taking $-\Delta$ as an isomorphism form $H_0^1(\Omega)$ to $H^{-1}(\Omega)$ is very useful, the spectrum properties are also important for an operator. However, we can only talk about the spectrum of an operator which maps a space to itself.

So we are going to talk about $-\Delta$ as an unbounded operator form $L^2(\Omega)$ to $L^2(\Omega)$. The first thing is to construct the domain of $-\Delta$. A direct way is to set $D(-\Delta) = H_0^2(\Omega)$, however $-\Delta$ is not a closed operator on this domain. An common way to extent $-\Delta$ to be a self-adjoint operator is to using the inner product introduced by $a(u, v) = (\nabla u, \nabla v)_{L^2}$ and using Friedrichs extension theory. Then you can have

Theorem 40. *The operator $-\Delta : L^2(\Omega) \to L^2(\Omega)$ is a self-adjoint operator with $D(-\Delta) = H^2(\Omega) \cap H_0^1(\Omega)$.*

In fact, we can construct the domain by simply analysing the inverse of $-\Delta$ from the regularity properties for elliptic operator. We have $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ is a isomorphism.So we can define the inverse $K = (-\Delta)^{-1}$ on $L^2(\Omega)$, rather than $H^{-1}(\Omega)$, in which case its range is a subspace of $H_0^1(\Omega)$.If the domain Ω is sufficiently smooth for elliptic regularity theory, then $u \in H^2(\Omega)$ if $f \in L^2(\Omega)$, and the range of K is $H^2(\Omega) \cap H_0^1(\Omega)$, for non-smooth domains, the range of K is more difficult to describe.

If we consider $-\Delta$ as an operator in $L^2(\Omega)$, then the domain of $-\Delta$ is $D(-\Delta) = Range(K)$, and we have:

$$-\Delta : D(-\Delta) \subset L^2(\Omega) \to L^2(\Omega)$$

is an unbounded self-adjoint linear operator with dense domain $H^2(\Omega) \cap H_0^1(\Omega)$.

The spectrum properties of $-\Delta$ will be expressed below:

Lemma 28. *If A is a compact self-adjoint operator from a infinite dimension Hilbert space to itself, then the spectrum of A denoted by $\sigma(A)$($\sigma_v(A)$ stand for the eigenvalues of A) satisfy the following properties.*

1. $r(A) = \lim_{n\to\infty} \|A^n\|^{\frac{1}{n}} = \|A\|$
2. $0 \in \sigma(A)$
3. $\sigma(A)\backslash\{0\} = \sigma_v(A)\backslash\{0\}$
4. *Follows one of the next cases:*
   - *a)* $\sigma(A) = \{0\} \Rightarrow A = 0$
   - *b)* $\sigma(A)\backslash\{0\}$ *is a finite set*
   - *c)* $\sigma(A)\backslash\{0\}$ *is a sequence with only one accumulation* 0.

Theorem 41. *The operator $-\Delta : L^2(\Omega) \to L^2(\Omega)$ has an increasing sequence of real eigenvalues of finite multiplicity*

$$0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

such that $\lambda_n \to \infty$, and $-\Delta e_i = \lambda_i e_i$ with $e_i \in H_0^1(\Omega) \cap C^\infty(\Omega)$ if Ω is C^∞ smoothness. What's more, $\{e_i\}_{i=1}^\infty$ is the Hilbert basis for $L^2(\Omega)$ even for $H_0^1(\Omega)$.

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[1] In fact, for any $f, g \in L^2$, let $u = (-\Delta)^{-1}f, v = (-\Delta)^{-1}g$, then $u, v \in H_0^1(\Omega)$ and $((-\Delta)^{-1}f, g) = -(u, \Delta v) = (\nabla u, \nabla v) = (-\Delta u, v) = (f, (-\Delta)^{-1}g)$.

Proof.

- First, we need find a uniform space to redefine $-\Delta$ to discuss its spectrum. We can do as mentioned before by define $-\Delta : L^2(\Omega) \to L^2(\Omega)$ as an unbounded operator.As we discussed before, Δ can been seen as a self-adjoint unbounded operator on $H^2(\Omega) \cap H_0^1(\Omega)$.
- It's difficult to analyz the spectrum of $-\Delta$ directly.Thanks to the Pioncare inequality(Garding inequality in general elliptic operator), we know that $(-\Delta)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is a self-adjoint positive definition compact operator.

  Self-adjoint: This can be checked that: for any $f, g \in L^2(\Omega)$,$(-\Delta^{-1}(f), g) = (u, -\Delta v) = (\nabla u, \nabla v) = (-\Delta u, v) = (f, -\Delta^{-1}(g))$

  Positive definition: For any $f \in L^2(\Omega)$ and$f \neq 0$, then $(-\Delta^{-1}(f), f) = (\nabla u, \nabla u) \geq C(u, u) > 0$.

  Compact: This is we put $-\Delta^{-1} : L^2(\Omega) \to H_0^1(\Omega) \to L^2(\Omega)$, thanks to the Sobolev compact embedding theory, $H_0^1(\Omega) \to L^2(\Omega)$ is a compact imbedding, so $-\Delta$ is a compact operator form $L^2(\Omega)$ to $L^2(\Omega)$.
- As $(-\Delta)^{-1}$ has those wonderful properties, we denote $K = (-\Delta)^{-1}$ and analyze this operator using the theorem above. Then we know that, there exist $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots$ with $\lim_{n\to\infty} \mu_n = 0$ with eigenfunctions $\{e_i\}_{i=1}^{\infty}$.
- Then we get $\lambda_i = \mu_i^{-1}$ for $-\Delta$ with $-\Delta e_i = \lambda_i e_i$, and $\{e_i\}_{i=1}^{\infty}$ forms a complete Hilbert basis for $L^2(\Omega)$. And those are the all eigenvalues and eigenfunctions of $-\Delta$.

□

5.3 On the spectrum of $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$

We consider the Dirichlet problem for the Laplace operator on a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. Here, Ω is with Lipschitz boundary.

We have the following Poincarè inequality: There exists a constant $C(\Omega)$ such that

$$\|u\|_0 \leq C|u|_1. \tag{5.10}$$

Poincarè inequality implies that there is a unique solution to the Dirichlet problem, which in weak form is: Find $u \in H_0^1(\Omega)$ such that

$$(\nabla u, \nabla v) = (f, v), \quad \text{for all} \quad v \in H_0^1(\Omega).$$

We note that this defines an unbounded linear operator $A : L^2 \mapsto L^2$ with a dense domain $H_0^1(\Omega) \subset L^2$. If u is the solution to $Au = f$ and $f \in L^2$ then we know that $u \in H_0^1(\Omega)$ is unique and satisfies

$$|u|_1 \leq |f|_0.$$

Hence,

$$|u|_1 \leq 1 \quad \text{for all} \quad f \quad \text{such that } \|f\|_0 = 1.$$

In accordance with this, let B be the operator that maps $f \in L^2(\Omega)$ to $u \in H_0^1(\Omega)$, that is the inverse of A, namely $u = Bf$. Note that $ABf = f$ and $BAu = u$ for all $f \in L^2(\Omega)$ and all $u \in H_0^1(\Omega)$.

Since the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact we have that the unit ball in $L^2(\Omega)$ is mapped by B to a set with compact closure (pre-compact set) in $L^2(\Omega)$.

It follows then that B is: (1) compact; (2) self-adjoint; and (3) positive definite operator.

Proof.

(1) **Compactness:** Compact operator is an operator which maps a bounded set into a pre-compact set (i.e. a set whose closure is compact). B maps the unit ball of L^2 to the set $S_1 = \{u : |u|_1 \le 1\}$ in H_0^1. The embedding theorem (Relich-Kondrashov Theorem) says that the embedding of H_0^1 into L^2 is compact. Equivalently, the inclusion operator $H_0^1(\Omega) \mapsto L^2(\Omega)$ is compact and maps S_1 (a bounded set in $H_0^1(\Omega)$) to a pre-compact set in L^2, that is S_1 is a pre-compact set in $L^2(\Omega)$ (set with compact closure). Since B maps the unit ball to S_1 it follows that B is compact.

(2) **Symmetry:** For any f and g there exist $u \in H_0^1$ and $v \in H_0^1$ such that $Au = f$ and $Av = g$. We then have $(Bg, f) = (BAu, Av) = (u, Av) = (Au, v)$. Further, we also have $(Au, v) = (ABf, Bg) = (f, Bg)$ and B is self-adjoint.

(3) **Positivity:** Same as symmetry, for any $f \in L^2$, we have $(Bf, f) = (BAu, Au) = (u, Au) \ge C(u, u) > 0$, and this shows that all eigenvalues are positive.

□

The fact that a compact self-adjoint operator has complete set of eigenvectors and eigenfunctions is known as the Hilbert-Schmidt theorem below (Theorem 42). So B has a complete orthonormal (in L^2) basis satisfying

$$B\phi_j = \mu_j\phi_j,$$

and $\mu_j \to 0$, when $j \to \infty$ and $\mu_j > 0$. Since B maps L^2 to H_0^1 it follows that $\phi_j \in H_0^1$.

For A then we have

$$\phi_j = AB\phi_j = \mu_j A\phi_j,$$

and hence all eigenfunctions of B are also eigenfunctions of A with eigenvalues $\lambda = \mu_j^{-1}$. Since this is a complete set in L^2 it follows that these are the ONLY eigenfuctions of A.

Theorem 42 (Hilbert-Schmidt Theorem). *Let $B : X \mapsto X$ be a compact self-adjoint operator on a Hilbert space X. Then the set of eigenfunctions of B is complete.*

Proof. Take the space spanned by the eigenfunctions $M = \text{span}\{\phi_k\}$. We want to show that the restriction of B on $M^\perp$ is zero. Let us call this restriction B_0. If it were not zero, this means $\|B_0\| > 0$,

Since we are dealing with self-adjoint compact operator, there exist an eigenvalue λ such that $\|B_0\| = \lambda$. But this means that B_0 has a nonzero eigenvalue and B_0 has eigenfunction in $M^\perp$. This should also be eigenfunction of B.

However, M was the space that contains ALL eigenfunctions. This means that $\|B_0\| = 0$ and we have that $BM^\perp = 0$.

But now, this implies that so $M^\perp$ is a subspace of the Kernel of B, i.e. $M^\perp$is a subspace of the eigenspace corresponding to the 0 eigenvalue, and hence $M^\perp \subset M$ (because M contains all eigenfunctions and contains the Kernel of B for sure if any kernel).

This can only happen if $M^\perp = 0$ and the proof is complete. □

5.3.1 On the positivity of the first eigenfunction of $-\Delta$

Theorem 43. *Let $\phi_1 \in H_0^1(\Omega)$ be the first eigenfunction of $-\Delta : H_0^1(\Omega) \mapsto H^{-1}(\Omega)$, namely*

$$-\Delta\phi_1 = \lambda_1\phi_1$$

where

$$\lambda_1 = \inf_{v \in H_0^1(\Omega)} \frac{\langle -\Delta v, v\rangle}{|v|_{1,\Omega}}.$$

Then

$$\phi_1(x) \neq 0 \quad x \in \Omega.$$

Proof. Let's assume without loss of generality that $|\phi_1|_{1,\Omega} = 1$. Since $\phi_1^\pm \in H_0^1(\Omega)$, with

$$\phi_1^+ = \phi_1 I(\{\phi_1 \geq 0\})$$

$$\phi_1^- = \phi_1 I(\{\phi_1 \leq 0\})$$

We have $\langle -\Delta\phi_1^+, \phi_1^- \rangle = 0$. Hence

$$\begin{aligned}\lambda_1 &= \langle -\Delta\phi_1, \phi_1 \rangle = \langle -\Delta\phi_1^+, \phi_1^+ \rangle + \langle -\Delta\phi_1^-, \phi_1^- \rangle \\ &\geq \lambda_1 |\phi_1^+|_{1,\Omega}^2 + \lambda_1 |\phi_1^-|_{1,\Omega}^2 \\ &= \lambda_1 |\phi_1|_{1,\Omega} = \lambda_1\end{aligned}$$

The inequality above must in fact be an equality, and so

$$\langle -\Delta\phi_1^+, \phi_1^+ \rangle = \lambda_1 |\phi_1^+|_{1,\Omega}^2, \ \langle -\Delta\phi_1^-, \phi_1^- \rangle = \lambda_1 |\phi_1^-|_{1,\Omega}^2.$$

Therefor ϕ_1^+ and ϕ_1^- are also eigenfunctions of λ_1. So

$$-\Delta\phi_1^+ = \lambda_1 \phi_1^+ \geq 0 \text{ in } \Omega$$

Thus the strong maximum principle implies either $\phi_1^+ > 0$ in Ω or else $\phi_1^+ \equiv 0$ in Ω. Similar arguments apply to ϕ_1^-. So $\phi_1(x) \neq 0$ in Ω.

□

5.3.2 On the positivity of $(-\Delta)^{-1}$

Theorem 44. *The inverse of Laplacian, $(-\Delta)^{-1}$ is non-negative operator in the sense that*

$$((-\Delta)^{-1} f)(x) \geq 0 \text{ as long as } f(x) \geq 0, \quad x \in \Omega.$$

Proof. We first prove the Maximum principle:

$$\text{if } -\Delta u \geq 0 \text{ in } \Omega, \text{ then } \min_{x \in \bar{\Omega}} u = \min_{x \in \partial\Omega} u.$$

For $\epsilon > 0$, define an auxiliary function $\tilde{u} = u - \epsilon x_1^2$. We have

$$-\Delta\tilde{u} = -\Delta u + 2\epsilon > 0 \text{ in } \Omega, \text{ and } \min_{x \in \bar{\Omega}} \tilde{u} = \min_{x \in \partial\Omega} \tilde{u}.$$

In fact, if there exists $x_0 \in \Omega$ such that $\min_{\bar{\Omega}} \tilde{u} = \tilde{u}(x_0)$, then, $D\tilde{u}(x_0) = 0$ and $D^2\tilde{u}(x_0) \geq 0$ (non-negative definite matrix). $-\Delta\tilde{u}(x_0) = -\sum_{i=1}^n \tilde{u}_{x_i x_i}(x_0) \leq 0$, which is a contradiction. And $\min_{x \in \bar{\Omega}} \tilde{u} = -\max_{x \in \bar{\Omega}} -\tilde{u}$,

$$|\max_{x \in \bar{\Omega}} -\tilde{u} - \max_{x \in \bar{\Omega}} -u| \leq |\max_{x \in \bar{\Omega}}(-\tilde{u} + u)| \to 0 \text{ as } \epsilon \to 0.$$

Thus, Let $\epsilon \to 0$, we get $\min_{x \in \bar{\Omega}} u = \min_{x \in \partial\Omega} u$. Therefore, by $-\Delta((-\Delta)^{-1} f)(x) = f(x) \geq 0$, we have

$$(-\Delta)^{-1} f(x) \geq \min_{x \in \bar{\Omega}} (-\Delta)^{-1} f = \min_{\partial\Omega} (-\Delta)^{-1} f = 0.$$

□

5.4 grad, curl and div operators, exact sequences

The Sobolev spaces $H(\mathrm{div})$ and $H(\mathrm{curl})$ are two important classes of spaces that are important for mixed formulation of second order elliptic equations and for Maxewell equations in electromagnetic applications. In this section, we give some brief discussions on finite element subspaces for these two spaces and the corresponding multigrid methods. [**?**]

Given a Lipschitz domain Ω and a linear differential operator D, we define

$$H(\mathrm{D};\Omega) = \{v \in (L^2(\Omega))^n, \mathrm{D}v \in L^2(\Omega)\}.$$

By taking $D = \mathrm{div}, \mathrm{curl}$, we obtain the Sobolev spaces $H(\mathrm{div};\Omega)$ and $H(\mathrm{curl};\Omega)$ that we are interested. We also note that

$$H^1(\Omega) = H(\mathrm{grad};\Omega), \quad L^2(\Omega) = H(0;\Omega).$$

In 3D, the grad, curl and div have the following formulations:

$$\mathrm{grad}u = \nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \\ \partial_z u \end{pmatrix}, \quad \mathrm{curl}\underset{\sim}{u} = \nabla \times \underset{\sim}{u} = \begin{pmatrix} \partial_y w - \partial_z v \\ \partial_z u - \partial_x w \\ \partial_x v - \partial_y u \end{pmatrix}, \quad \mathrm{div}\underset{\sim}{u} = \nabla \cdot \underset{\sim}{u} = \partial_x u + \partial_y v + \partial_z w.$$

Lemma 29. *A vector function* $\underset{\sim}{u} \in L^2(\Omega)$ *belongs to* $H(\mathrm{curl})$ *(or* $H(\mathrm{div})$*) if*

1. $\underset{\sim}{u}$ *is piecewise smooth with respect to a partition of* Ω*, and*
2. $\underset{\sim}{u} \times \underset{\sim}{n}$ *(or* $\underset{\sim}{u} \cdot \underset{\sim}{n}$*) is continuous across the boundaries of the subdomains in the partition.*

Proof. Consider $\Omega = \Omega_1 \cup \Omega_2$ and $\Gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$. Then, for any $\underset{\sim}{\chi} \in (C_0^\infty(\Omega))^n$

$$\int_{\Omega_i} \underset{\sim}{\chi} \cdot \mathrm{curl}\underset{\sim}{u} - \int_{\Omega_i} \mathrm{curl}\underset{\sim}{\chi} \cdot \underset{\sim}{u} = -\int_\Gamma (\underset{\sim}{u} \times \underset{\sim}{n}) \cdot \chi, \quad i = 1, 2.$$

Since $\underset{\sim}{u} \times \underset{\sim}{n}$ is continuous across Γ, we have

$$\int_{\Omega_1 \cup \Omega} \underset{\sim}{\chi} \cdot \mathrm{curl}\underset{\sim}{u} = \int_\Omega \mathrm{curl}\underset{\sim}{\chi} \cdot \underset{\sim}{u}.$$

Hence $\underset{\sim}{u} \in H(\mathrm{curl})$. A similar argument follows for the $H(\mathrm{div})$. □

If Ω is simply connected, we have

$$\mathrm{curl}\underset{\sim}{u} = 0 \Leftrightarrow \underset{\sim}{u} = \mathrm{grad}\phi,$$

$$\mathrm{div}\underset{\sim}{u} = 0 \Leftrightarrow \underset{\sim}{u} = \mathrm{curl}\phi.$$

Next we introduce the exact sequence

$$\mathbb{R} \xrightarrow{\subset} C^\infty(\Omega) \xrightarrow{\mathrm{grad}} C^\infty \xrightarrow{\mathrm{curl}} C^\infty \xrightarrow{\mathrm{div}} C^\infty \longrightarrow 0. \tag{5.11}$$

"Exact sequence" here means that for each right arrow, the range of the left operator equals the kernel of the right operator, namely,

1. $\mathbb{R} = N(\mathrm{grad})$;
2. *Range*(grad) = N(curl);
3. *Range*(curl) = N(div);
4. *Range*(div) = $C^\infty(\Omega)$.

de Rham complex

The de Rham complex can be presented in a general framework by means of exterior calculus. Let us suppose that $\Omega \subset \mathbb{R}^3$ is a simply connected domain. A direct calculation shows that if $p \in H^1(\Omega)$ then $\nabla p \in H(\mathrm{curl}, \Omega)$ since $\nabla \times \nabla p = 0 \in (L^2(\Omega))^3$ and $\nabla p \in (L^2(\Omega))^3$. Similarly, if $\underset{\sim}{u} \in H(\mathrm{curl}; \Omega)$, then $\nabla \times \underset{\sim}{u} \in H(\mathrm{div}; \Omega)$. These results, and the corresponding results for the divergence applied to functions in $H(\mathrm{div}; \Omega)$, can be summarized in the following de Rham diagram in 3D case:

$$\mathbb{R} \xrightarrow{\subset} H^1(\Omega) \xrightarrow{\mathrm{grad}} H(\mathrm{curl}, \Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div}, \Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \longrightarrow 0. \tag{5.12}$$

Let us derive the 2D de Rham complexes from the 3D one. For the 2D case, all the variables should only depend on x and y. Then for any vector function $\underset{\sim}{u}$,

$$\nabla \times \underset{\sim}{u} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \begin{pmatrix} \partial_y w \\ -\partial_x w \end{pmatrix} \\ \partial_x v - \partial_y u \end{pmatrix} = \begin{pmatrix} \mathrm{curl} w \\ \mathrm{rot} \begin{pmatrix} u \\ v \end{pmatrix} \end{pmatrix},$$

where the two dimensional curl and rot are defined as

$$\mathrm{curl} w = (\partial_y w, -\partial_x w)^T, \quad \mathrm{rot} \underset{\sim}{u} = \partial_x v - \partial_y u.$$

In this case, the rot operator can be understood as $\mathrm{rot}\underset{\sim}{u} = \mathrm{div}(\underset{\sim}{u}^\perp)$, where $\underset{\sim}{u}^\perp$ is the rotation of $\underset{\sim}{u}$ by an angle of $\pi/2$. Let

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Then

$$\mathrm{rot} \begin{pmatrix} u \\ v \end{pmatrix} = \partial_x v - \partial_y u = \mathrm{div} \begin{pmatrix} v \\ -u \end{pmatrix} = \mathrm{div} U \begin{pmatrix} u \\ v \end{pmatrix},$$

$$\mathrm{curl} w = \begin{pmatrix} \partial_y w \\ -\partial_x w \end{pmatrix} = U \begin{pmatrix} \partial_x w \\ \partial_y w \end{pmatrix} = U \mathrm{grad} w.$$

In other words, rot = div$U$ and curl = Ugrad.

For any $u = u(x, y)$,

$$\mathrm{rot}\ \mathrm{grad} u = \mathrm{rot} \begin{pmatrix} \partial_x u \\ \partial_y u \\ 0 \end{pmatrix} = \begin{pmatrix} -\partial_{yz} u \\ \partial_{xz} u \\ \partial_{xy} u - \partial_{xy} u \end{pmatrix} = 0, \tag{5.13}$$

and

$$\text{div curl}u = \text{div}\begin{pmatrix} \partial_y u \\ -\partial_x u \\ \partial_x u - \partial_y u \end{pmatrix} = \partial_{xy}u - \partial_{xy}u + (\partial_{xz}u - \partial_{yz}u) = 0. \tag{5.14}$$

Hence, we have two de Rham complexes in 2D:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\subset"] & H^1(\Omega) \arrow[r, "\text{grad}"] & H(\text{rot},\Omega) \arrow[r, "\text{rot}"] & L^2(\Omega) \arrow[r] & 0. \\ \mathbb{R} \arrow[r, "\subset"] & H^1(\Omega) \arrow[r, "\text{curl}"] & H(\text{div},\Omega) \arrow[r, "\text{div}"] & L^2(\Omega) \arrow[r] & 0. \end{tikzcd} \tag{5.15}$$

Theorem 45. *The diagrams* (5.12), (5.15) *and have the property that the range of one operator is contained in the kernel of the one following it in the sequence.*

The continuous case of the exact sequences is a special case of the Poincaré Lemma, which states that if $\Omega \subseteq \mathbb{R}^n$ is a constractible domain, and k is an integer, then for any k-form ω such that $d\omega = 0$, there exists a $k-1$-form such that $\omega = d\alpha$ (In other words, all closed differential k-forms on contractible domains are exact). The proof of Poincaré Lemma based on counting dimensions of the cohomology on a contractible domain.

Another way of proving the exactness of the de Rham sequences is constructive.

5.5 H(grad), H(curl) and H(div) spaces

We can define spaces

$$H(D;\Omega) = \left\{v \in L^2(\Omega),\ Dv \in L^2(\Omega)\right\},$$

where

$$D = \begin{cases} \text{grad}, \\ \text{rot}, \\ \text{curl}, \\ \text{div}, \end{cases} \quad \text{if } \Omega \subset \mathbb{R}^2, \quad D = \begin{cases} \text{grad}, \\ \text{curl}, \\ \text{div}, \end{cases} \quad \text{if } \Omega \subset \mathbb{R}^3.$$

Here, we abuse the notation a little bit, namely, we use $L^2(\Omega)$ to denote $L^2(\Omega)$, $[L^2(\Omega)]^2$ or $[L^2(\Omega)]^3$. Furthermore, we can define space $H_0(D;\Omega)$

$$H_0(D;\Omega) = \{v \in H(D;\Omega),\ \text{tr } v = 0\},$$

where tr is the trace operator, which is defined by

$$\text{tr } v = \begin{cases} v,\ v \in H(\text{grad};\Omega), \\ v,\ v \in H(\text{curl};\Omega), \\ \tau \cdot v,\ v \in H(\text{rot};\Omega), \\ \nu \cdot v,\ v \in H(\text{div};\Omega), \end{cases} \quad \text{if } \Omega \subset \mathbb{R}^2. \quad \text{tr } v = \begin{cases} v,\ v \in H(\text{grad};\Omega), \\ \nu \times v,\ v \in H(\text{curl};\Omega), \\ \nu \cdot v,\ v \in H(\text{div};\Omega), \end{cases} \quad \text{if } \Omega \subset \mathbb{R}^3.$$

Here, τ is the tangential direction along the edge, while ν is the normal direction of the edge (in 2D) or face (in 3D).

Similarly, we can define $H_{0,\Gamma_D}(D;\Omega)$

$$H_{0,\Gamma_D}(D;\Omega) = \{v \in H(D;\Omega),\ \mathrm{tr}_{\Gamma_D} v = 0\}.$$

Useful identities

For a bounded Lipschitz domain Ω, the outer normal vector $\nu = (\nu_i)$ of $\partial\Omega$ can be well defined almost everywhere on $\partial\Omega$. The following identities will be frequently used in this book.

$$\int_\Omega (D_i u) v dx = -\int_\Omega u D_i v dx + \int_{\partial\Omega} u v \nu_i ds. \tag{5.16}$$

$$\int_\Omega \nabla u \cdot \nabla v dx = -\int_\Omega (\Delta u) v dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v ds. \tag{5.17}$$

$$\int_\Omega \mathrm{div}\ w dx = \int_{\partial\Omega} w \cdot \nu ds \tag{5.18}$$

$$\int_\Omega \nabla \times u \cdot v dx = \int_{\partial\Omega} (\nu \times u) \cdot v ds + \int_\Omega u \cdot \nabla v dx. \tag{5.19}$$

Equations (5.21) - (5.23) hold for both 2D and 3D, while equation (5.19) only hold in 3D. In 2D, the $\nabla\times$, i.e. the curl operator, degenerates to the rot operator. So the integration by parts formula becomes

$$\int_\Omega \mathrm{rot} u \cdot v dx = \int_{\partial\Omega} (\nu \times u) v ds + \int_\Omega u \cdot \mathrm{curl} v dx, \tag{5.20}$$

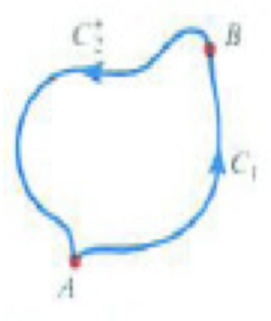
where $\mathrm{curl} v = \left(\frac{\partial v}{\partial y}, -\frac{\partial v}{\partial x}\right)^T$ (for any scalar function v), and $\mathrm{rot} u = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}$ (for any $u = (u_1, u_2)^T$). And for $\nu = (\nu_1, \nu_2)^T$, $u = (u_1, u_2)^T$, the cross product in 2D is defined as $\nu \times u = \nu_1 u_2 - \nu_2 u_1$.

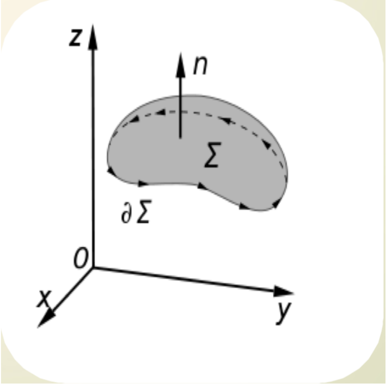
5.6 Stoke's theorem and Trace theorems

Stoke's theorem

Stokes' theorem (also called the Stokes–Cartan theorem) is a statement about the integration of differential forms on manifolds. It simplifies and generalizes several theorems from vector calculus. Stokes' theorem says that the integral of a differential form ω over the boundary of some orientable manifold Ω is equal to the integral of its exterior derivative $d\omega$ over the whole of Ω, i.e.,

$$\int_{\partial\Omega} \omega = \int_\Omega d\omega.$$





1. d = grad: for a curve C begins with node a and ends with node b, stoke's theorem reads

$$\int_C \mathrm{grad}u\cdot \underset{\sim}{t} = \int_{\{a,b\}} u = u(b) - u(a).$$

2. d = curl: stoke's theorem reads

$$\int_S \mathrm{curl}\underset{\sim}{u}\cdot \underset{\sim}{n} = \int_{\partial S} \underset{\sim}{u}\cdot \underset{\sim}{t}.$$

3. d = div: stoke's theorem reads

$$\int_V \mathrm{div}\underset{\sim}{u} = \int_{\partial V} \underset{\sim}{u}\cdot \underset{\sim}{n}.$$

Green's identities

For a bounded Lipschitz domain Ω, the outer normal vector $\nu = (\nu_i)$ of $\partial\Omega$ can be well defined almost everywhere on $\partial\Omega$. The following identities will be frequently used in this book.

$$\int_\Omega (D_i u) v dx = -\int_\Omega u D_i v dx + \int_{\partial\Omega} u v \nu_i ds. \tag{5.21}$$

$$\int_\Omega \nabla u\cdot\nabla v dx = -\int_\Omega (\Delta u) v dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v.ds \tag{5.22}$$

$$\int_\Omega \mathrm{div}\ w dx = \int_{\partial\Omega} w\cdot \nu ds. \tag{5.23}$$

Trace theorems

The particular useful case of the above theorem is when $p = 2$. Because of its extraordinary importance, let us restate the theorem and some its consequence.

Theorem 46. *1. The mapping $u \to \gamma u \equiv u|_{\partial\Omega}$ which is defined for $u \in C^1(\bar{\Omega})$ has a unique continuous extention as an operator from $H^1(\Omega)$ onto $H^{1/2}(\partial\Omega)$.*

2. *The operator* $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ *has a right continuous inverse, namely there exists a constant* c_0 *such that for any* $f \in H^{1/2}(\partial\Omega)$*, there corresponds to a function* $v_f \in H^1(\Omega)$ *such that*
$$f = \gamma v_f \quad \textit{and} \quad \|v_f\|_{1,\Omega} \le c_0\|f\|_{1/2,\partial\Omega}.$$
3. *For any* $u \in H^1(\Omega)$*,*
$$\|u\|_{\frac{1}{2},\partial\Omega} \eqsim \inf_{v\in u+H_0^1(\Omega)} \|v\|_{1,\Omega} \le \|u\|_{1,\Omega}.$$

For smooth domains, the above trace theorem and its proof can be found in most standard text books on Sobolev spaces. But for Lipschitz domains, the proof is much less well-known and it can be found in Necas [..].

The hint is given in the following theorem.

Theorem 47. *Assume that* K_1 *and* K_2 *are Lipchitz domain,* $\Sigma = K_1 \cap K_2$*, and function* $u(x) \in \mathbb{R}^d$ *(*$d = 1, 2$ *or* 3*) is given by:*
$$u(x) = \begin{cases} u_1(x), & x \in K_1 \\ u_2(x), & x \in K_1 \end{cases}$$

1. $u \in H^1(K_1 \cup K_2)$ *iff* u *is continuous across* Σ*.*
2. $\mathbf{u} \in H(curl, K_1 \cup K_2)$ *iff* $\mathbf{u_1} \times \boldsymbol{n} = \mathbf{u_2} \times \boldsymbol{n}$*, where* $\boldsymbol{n}$ *is the normal vector of* Σ *pointing from* K_1 *to* K_2*.*
3. $\mathbf{u} \in H(div, K_1 \cup K_2)$ *iff* $\mathbf{u_1} \cdot \boldsymbol{n} = \mathbf{u_2} \cdot \boldsymbol{n}$*, where* $\boldsymbol{n}$ *is the normal vector of* Σ *pointing from* K_1 *to* K_2*.*

Proof. Here we only give a proof of the second case. Proof of the first case can be found in the book of Ciarlet, and proof of the third case is similar to the second one. By integration by parts, we get:
$$\int_{K_1\cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1\cup K_2} \mathbf{u} \cdot \mathrm{curl}\phi, \ \forall \phi \in C_0^\infty(K_1 \cup K_2)$$

By Stokes theorem:
$$\int_{K_1\cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1} \mathbf{u_1} \cdot \mathrm{curl}\phi + \int_{K_2} \mathbf{u_2} \cdot \mathrm{curl}\phi + \int_{\Sigma} (\mathbf{u_1} \times \mathbf{n_1} + \mathbf{u_2} \times \mathbf{n_2}) dA$$

and $\mathbf{n_1} = \mathbf{n}$, $\mathbf{n_2} = -\mathbf{n}$, so:
$$\int_{K_1\cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1} \mathbf{u_1} \cdot \mathrm{curl}\phi + \int_{K_2} \mathbf{u_2} \cdot \mathrm{curl}\phi$$

Furthermore,
$$\|curl\mathbf{u}\|^2_{L^2(K_1\cup K_2)} = \|curl\mathbf{u_1}\|^2_{L^2(K_1)} + \|curl\mathbf{u_2}\|^2_{L^2(K_2)}$$

□

There are several important remarks about the above theorem. H^1-conforming elements are continuous for any dimension ($d = 1, 2, 3$). $H(curl)$-conforming elements have continuous tangential components across the boundary, while $H(div)$-conforming elements have continuous normal components across the boundary.

Theorem 48. *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain in* $\mathbb{R}^3$ *with unit outward normal* ν*. Then*

1. *the mapping* $u \mapsto \gamma u = \nu \times u \,|_{\partial\Omega}$*, which is defined for* $u \in [C^\infty(\bar{\Omega})]^3$*, can be extended by continuity to a continuous linear map* γ *from* $H(\mathrm{curl}; \Omega)$ *to* $H^{-1/2}(\partial\Omega)$*. Therefore,*
$$\|\nu \times u\|_{H^{-1/2}(\partial\Omega)} \le C\|u\|_{H(\mathrm{curl};\Omega)}.$$

2. *the following Green's theorem holds for any* $v \in H(\mathrm{curl};\Omega)$ *and* $\phi \in [H^1(\Omega)]^3$

$$(\nabla \times v, \phi) - (v, \nabla \times \phi) = \langle \gamma v, \phi \rangle_{\partial\Omega}.$$

Remark 13. The map $\gamma : \ H(\mathrm{curl};\Omega) \rightarrow [H^{-1/2}(\partial\Omega)]^3$ is not surjective since for any v, the trace map γv is tangential to $\partial\Omega$, whereas $[H^{-1/2}(\partial\Omega)]^3$ contains vectors that are not tangential to $\partial\Omega$.

Theorem 49. *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded Lipschitz domain in* $\mathbb{R}^3$ *with unit outward normal* ν. *Then*

1. *the mapping* $u \mapsto \gamma u = u \mid_{\partial\Omega} \cdot \nu$, *which is defined for* $u \in [C^\infty(\bar{\Omega})]^3$, *can be extended by continuity to a continuous linear map* γ *from* $H(\mathrm{div};\Omega)$ *onto* $H^{-1/2}(\partial\Omega)$. *Therefore,*

$$\|v \cdot \nu\|_{H^{-1/2}(\partial\Omega)} \leq C\|v\|_{H(\mathrm{div};\Omega)}.$$

2. *the following Green's theorem holds for functions* $v \in H(\mathrm{div};\Omega)$ *and* $\phi \in H^1(\Omega)$

$$(v, \nabla\phi) + (\nabla \cdot v, \phi) = \langle \phi, \gamma v \rangle_{\partial\Omega}.$$

5.7 Poincare ineqaulity

[c1]

[c1] Simo can add things to this section. Add the 'generalized Poincarè inequality and its proof' here

Theorem 50 (Poincare inequality). *if* Ω *is a bounded domain, for arbitrary* $v \in H_0^1(\Omega)$*:*

$$\|v\|_{0,\Omega} \leq C\|\nabla v\|_{0,\Omega}$$

Note that if $v \mid_{\partial\Omega} \neq 0$, this inequality is wrong.

Proof. Notice that $\partial_i(x_i v^2) = v^2 + 2x_i v \partial_i v$, therefore,

$$\begin{aligned}
&\int_\Omega v^2 \\
&= \int_\Omega \partial_i(v^2 x_i) - \int_\Omega 2x_i v \partial_i v \\
&= -2\int_\Omega x_i v \partial_i v \\
&\leq C\int_\Omega \mid v \mid \cdot \mid \partial_i v \mid \quad \text{(Cauchy-Schwarz inequality)} \\
&\leq C\|v\|_{0,\Omega} \cdot \|\partial_i v\|_{0,\Omega}
\end{aligned}$$

So $\|v\|_{0,\Omega} \leq C\|\partial_i v\|_{0,\Omega}$. ☐

Actually, we can change the condition of vanishing on the boundary into an averaged type in the above theorem, and the result still holds.

Theorem 51 (Poincare inequality(averaged type)). *if* Ω *is a bounded convex domain, for arbitrary* $v \in H_0^1(\Omega)$ *and* $\int_\Omega v = 0$*:*

$$\|v - \overline{v}\|_{0,\Omega} \leq \frac{C_d d_\Omega^{d+1}}{\mid \Omega \mid}\|\nabla v\|_{0,\Omega}$$

Where d_Ω *is the diameter of* Ω*,* $\mid \Omega \mid$ *is the area of* Ω*,* $\overline{v} = \frac{1}{\Omega}\int_\Omega v$

Proof. First, let us consider this equality,which can be easily examined by direct computation:

$$| \Omega |^2 \|v - \hat{v}\|_{0,\Omega} = \int_\Omega | \int_\Omega v(x) - v(y) |^2 \ (1)$$

By fundamental theorem of calculus, we have:

$$\int_\Omega v(y) - v(x) dy = \int_\Omega dy \int_0^1 \nabla v(x + t(y - x)) \cdot (y - x) dt$$

We introduce another variable $z = x + t(y - x)$ so for a given x ∈ Ω , (y,t) ∈ Ω×(0,1] is equivalent to (z,t) ∈ G Where $G = (z, t), t \in (0, 1], \frac{(z-x)}{t} + x \in \Omega$, and notice that G ⊂ $\Omega \times (0, 1]$ So by change of variable,

$$\int_\Omega v(y) - v(x) dy = \int_G \nabla v(z) \cdot \frac{z - x}{t^{d+1}} dt dz$$

$$= \int_\Omega \int_0^1 \kappa_G \cdot \nabla v(z) \cdot \frac{z - x}{t^{d+1}} dt$$

Because$(z, t) \in G$, so $\frac{|z-x|}{t} < d_\Omega$,then we get:

$$| \int_\Omega v(y) - v(x) dy | \leq \int_\Omega | \nabla v \| z - x | dz \int_{\frac{z-x}{d_\Omega}}^1 t^{-1-d} dt$$

$$= \frac{d_\Omega^d}{d} \int -\Omega | \nabla v \| z - x |^{1-d} dz (2)$$

And by Cauchy-Schwartz inequality, we have:

$$\int_\Omega (\int_\Omega | \nabla v \| z - x |^{1-d} dz)^2 dx \leq \int_\Omega (\int_\Omega | \nabla v |^2 | z - x |^{1-d} dz \int_\Omega | z - x |^{1-d}) dx$$

For any x∈ Ω, letB(x,d_Ω):=z∈ Ω :$| z - x | \leq d_\Omega$,then we have:

$$\int_\Omega | z - x |^{1-d} dz \leq \int_{B(x,d_\Omega)} | z - x |^{1-d} dz = \int_0^{d_\Omega} dC_d dr = dC_d d_\Omega$$

So,$\int_\Omega (\int_\Omega | \nabla v \| z - x |^{1-d} dz)^2 dx \leq dC_d d_\Omega \int_\Omega \int_\Omega | \nabla v |^2 | z - x |^{1-d} dz dx$ (3)

$$\leq (dC_d d_\Omega)^2 | v |_{1,\Omega}^2$$

So, by (1),(2) and(3),we have:

$$| \Omega |^2 \|v - \hat{v}\|_{0,\Omega} \leq C_d^2 d_\Omega^{2d+2} | v |_{1,\Omega}^2$$

Which is the desired result. □

Poincare inequality for other cases.

Theorem 52 (Poincare inequality). *If Ω is a bounded domain,*

1. *In 2D case,*
   a) *for arbitrary $v \in H_0(\mathrm{rot}; \Omega) \cap kernel(\mathrm{rot})^\perp$, there exists a constant C such that*

$$\|v\| \leq C\|\mathrm{rot}\ v\|.$$

b) for arbitrary $v \in H_0(\mathrm{curl};\Omega)$*, there exists a constant* C *such that*

$$\|v\| \le C\|\mathrm{curl}\ v\|.$$

c) for arbitrary $v \in H_0(\mathrm{div};\Omega) \cap kernel(\mathrm{div})^{\perp}$*, there exists a constant* C *such that*

$$\|v\| \le C\|\mathrm{div}\ v\|.$$

2. In 3D case,
a) for arbitrary $v \in H_0(\mathrm{curl};\Omega) \cap kernel(\mathrm{curl})^{\perp}$*, there exists a constant* C *such that*

$$\|v\| \le C\|\mathrm{curl}\ v\|.$$

b) for arbitrary $v \in H_0(\mathrm{div};\Omega) \cap kernel(\mathrm{div})^{\perp}$*, there exists a constant* C *such that*

$$\|v\| \le C\|\mathrm{div}\ v\|.$$