19

Differential Form

19.1 Three differential operators in 3D

Consider the differential form $\Omega \in \mathbb{R}^3$. For a scalar function

$$u : \Omega \to \mathbb{R}^1.$$

$$\text{grad}u = \begin{pmatrix} \frac{\partial u}{\partial x_1} \\ \frac{\partial u}{\partial x_2} \\ \frac{\partial u}{\partial x_3} \end{pmatrix}, \quad \text{grad} = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_3} \end{pmatrix} = \nabla.$$

Vector function $\mathbf{u} = (u_1, u_2, u_3)$.

$$\textbf{curl } \mathbf{u} = \begin{pmatrix} \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \\ \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\ \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \end{pmatrix} = \nabla \times \mathbf{u}$$

$$\mathrm{d}iv\mathbf{u} = \sum_{i=1}^{3} \partial_i u_i = \nabla \cdot \mathbf{u}.$$

curl grad: $\nabla \times (\nabla u) = 0$. div grad: $\nabla \cdot (\nabla \times \mathbf{u}) = 0$. So, if ω does not have holes,

$$Range(grad) \subset Ker(curl), \quad Range(curl) \subset Ker(div).$$

The three differential operators are each connected to an integral form by this Stokes Theorem:

$$\int_M dw = \int_{\partial M} w.$$

With respect to the operators, the specific form is:

1. d = grad, M = C (circle),

$$\int_C \nabla u \cdot ds = \int_{\partial C} u$$

2. d = curl, M = S (surface),

$$\iint_S (\nabla \times \underset{\sim}{v}) ds = \int_{\partial S} \underset{\sim}{v} \cdot \mathbf{t}$$

3. d = div, M = Ω (volume),

$$\iiint_\Omega \nabla \cdot \underset{\sim}{v} = \iint_{\partial\Omega} \underset{\sim}{v} \cdot \mathbf{n}$$

Define,

$$H(D;\omega) = \{v \in L^2(\Omega) | Dv \in L^2(\Omega)\},$$

where $D = grad, curl, div$. We have the exact sequence:

$$R \longrightarrow H(\text{grad}) \xrightarrow{\text{grad}} H(\text{curl}) \xrightarrow{\text{curl}} H(\text{div}) \xrightarrow{\text{div}} L^2 \longrightarrow 0$$

Exact sequence: If the range of any operator in the sequence is equal to the kernel of the operator on the right.

The exact sequence is one of the most important properties, as the kernel of the three operators then each has an exact description.

Finite element spaces for the H^1, $H(div)$ and $H(curl)$ spaces.

For $k = 0$, the $H_h^1(\Omega)$ is continuous piecewise linear finite element spaces and $L_h^2(\Omega)$ is the piecewise constant finite element space.

The space H_h^{curl}: For H_h^{curl}, on each element K, the shape function space is 6-dimensional

$$\mathcal{P} = \{\alpha + \beta \times x : \alpha, \beta \in \mathbb{R}^3\}$$

and the degrees of freedoms are tangential integral on each element edge

$$\mathcal{N} = \left\{ \int_e v \cdot t : \text{for each edge } e \subset K \right\}.$$

The space H_h^{div}: For $k = 0$ the shape function space for H_h^{div} on each element K is 4-dimensional

$$\mathcal{P} = \{\alpha + \beta x : \alpha \in \mathbb{R}^3, \beta \in \mathbb{R}\}$$

and the degrees of freedoms are normal integral on each element face

$$\mathcal{N} = \left\{ \iint_F v \cdot n : \text{ for each face } F \subset K \right\}.$$

The finite elements are summarised in the figure below:

An important kind of finite element space is conforming finite element. If W is a given space of functions, then V_h is W-conforming if $V_h \subseteq W$. A natural question is how to get conforming global finite element space if we know how to construct local finit element spaces. The hint is given in the following theorem.

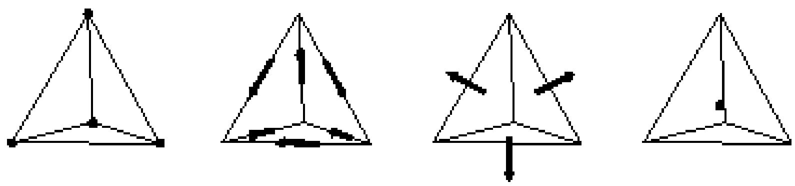

Fig. 19.1. Degrees of freedom for $H_h^1, H_h^{\text{curl}}, H_h^{\text{div}}$ and L_h^2 for $k = 0$.

Theorem 121. *Assume that K_1 and K_2 are Lipchitz domain, $\Sigma = K_1 \cap K_2$, and function $u(x) \in \mathbb{R}^d$ ($d = 1, 2$ or 3) is given by:*

$$u(x) = \begin{cases} u_1(x), & x \in K_1 \\ u_2(x), & x \in K_1 \end{cases}$$

1. *$u \in H^1(K_1 \cup K_2)$ iff u is continuous across Σ.*
2. *$\mathbf{u} \in H(curl, K_1 \cup K_2)$ iff $\mathbf{u_1} \times \boldsymbol{n} = \mathbf{u_2} \times \boldsymbol{n}$, where $\boldsymbol{n}$ is the normal vector of Σ pointing from K_1 to K_2.*
3. *$\mathbf{u} \in H(div, K_1 \cup K_2)$ iff $\mathbf{u_1} \cdot \boldsymbol{n} = \mathbf{u_2} \cdot \boldsymbol{n}$, where $\boldsymbol{n}$ is the normal vector of Σ pointing from K_1 to K_2.*

Proof. Here we only give a proof of the second case. Proof of the first case can be found in the book of Ciarlet, and proof of the third case is similar to the second one. By integration by parts, we get:

$$\int_{K_1 \cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1 \cup K_2} \mathbf{u} \cdot \text{curl}\phi, \ \forall \phi \in C_0^\infty(K_1 \cup K_2)$$

By Stokes theorem:

$$\int_{K_1 \cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1} \mathbf{u_1} \cdot \text{curl}\phi + \int_{K_2} \mathbf{u_2} \cdot \text{curl}\phi + \int_{\Sigma} (\mathbf{u_1} \times \mathbf{n_1} + \mathbf{u_2} \times \mathbf{n_2}) dA$$

and $\mathbf{n_1} = \mathbf{n}$, $\mathbf{n_2} = -\mathbf{n}$, so:

$$\int_{K_1 \cup K_2} curl\mathbf{u} \cdot \phi = \int_{K_1} \mathbf{u_1} \cdot \text{curl}\phi + \int_{K_2} \mathbf{u_2} \cdot \text{curl}\phi$$

Furthermore,

$$\|curl\mathbf{u}\|^2_{L^2(K_1 \cup K_2)} = \|curl\mathbf{u_1}\|^2_{L^2(K_1)} + \|curl\mathbf{u_2}\|^2_{L^2(K_2)}$$

□

There are several important remarks about the above theorem. H^1-conforming elements are continuous for any dimension ($d = 1, 2, 3$). $H(curl)$-conforming elements have continuous tangential components across the boundary, while $H(div)$-conforming elements have continuous normal components across the boundary.

For the finite elements aforementioned, the nodal parameters, together with the shape functions, guarantees the conformity of the finite element space respectively.

19.2 Exterior derivatives

19.2.1 k−vectors in 3D

Here we give intuitive explanation on differential forms rather than giving accurate mathematical definitions. Any vector $\xi \in \mathbb{R}^3$ is uniquely determined by magnititude and direction. We call such vector 1-vector.

Similarly, we can define 2-vector. Let's look at a special case first. Assume that

$$\mathbf{e_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ \mathbf{e_2} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \ \mathbf{e_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Then $\mathbf{e_i} \wedge \mathbf{e_j}$ $(i = 1, 2, 3; \ j = 1, 2, 3; \ i \neq j)$ forms a 2-vector. And it satisfies:

$$\mathbf{e_i} \wedge \mathbf{e_j} = -\mathbf{e_j} \wedge \mathbf{e_i}$$
$$\mathbf{e_i} \wedge \mathbf{e_i} = 0$$

In general, a 2-vector is formed by two 1-vectors, which is also charaterized by its magnitude and direction. For example, $\mathbf{u} \wedge \mathbf{v}$ is a 2-vector, whose magnitude is the area of parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$, and whose direction is determined by orientation. Assume that

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e_i}, \ \mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e_j}$$

Therefore,

$$\mathbf{u} \wedge \mathbf{v} = (\sum_{i=1}^{3} u_i \mathbf{e_i}) \wedge (\sum_{j=1}^{3} v_j \mathbf{e_j}) = \sum_{i,j} u_i v_j \mathbf{e_i} \wedge \mathbf{e_j}$$

The case of 3-vector is quite similar to 2-vector. For example, $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ is a 3-vector, whose magnitude is the volume of parallelpiped formed by $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$, and whose direction is determined by orientation. Assume that

$$\mathbf{u} = \sum_{i=1}^{3} u_i \mathbf{e_i}, \ \mathbf{v} = \sum_{j=1}^{3} v_j \mathbf{e_j}, \ \mathbf{w} = \sum_{k=1}^{3} w_j \mathbf{e_k}$$

Therefore,

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = (\sum_{i=1}^{3} u_i \mathbf{e_i}) \wedge (\sum_{j=1}^{3} v_j \mathbf{e_j}) \wedge (\sum_{k=1}^{3} w_j \mathbf{e_k}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) \mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$$

There is something interesting with determinant. In fact, determinant is defined via wedge product in a similar way to 3-vector in $\mathbb{R}^3$, i.e. assume that matrix $A = (\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n})$, then:

$$\det(A) = \det(\mathbf{a_1}, \mathbf{a_2}, \cdots, \mathbf{a_n}) = *(\mathbf{a_1} \wedge \mathbf{a_2} \wedge \cdots \wedge \mathbf{a_n})$$

19.2.2 Exterior derivative

After learning differential forms, we can define exterior derivative acting on differential forms. We still work on $\mathbb{R}^3$ in this section. And we will begin with 0-vector.

0-vector is the so-called 0-form. Assume that f is f is 0-form then f is a map from $\mathbb{R}^3$ to $\mathbb{R}$, and

$$d_0 f = \sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \mathbf{e_i} = \sum_{i=1}^{3} (\nabla f)_i \mathbf{e_i}$$

With the definition of exterior derivation for 0-form, we can define exterior derivative for higher order differential forms recursively. We will elaborate those cases one by one. For a 1-form $\mathbf{u} = \sum_{i=1}^{3} u_i(x) \mathbf{e_i}$, exterior derivative d_1 is defined as:

$$d_1\mathbf{u} = \sum_{i=1}^{3} d_1(u_i\mathbf{e_i}) = \sum_{i=1}^{3}(d_0u_i) \wedge \mathbf{e_i}$$

Notice that $d_0u_i = \sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\mathbf{e_j}$, so

$$\begin{aligned}
d_1\mathbf{u} &= \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\mathbf{e_j} \wedge \mathbf{e_i} \\
&=(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3})\mathbf{e_2} \wedge \mathbf{e_3} + (\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1})\mathbf{e_3} \wedge \mathbf{e_1} + (\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2})\mathbf{e_1} \wedge \mathbf{e_2} \\
&= \sum_{i=1}^{3}(\nabla \times \mathbf{u})_i(*\mathbf{e_i})
\end{aligned}$$

Similarly, for a 2-vector $\mathbf{u} = \sum_{i=1}^{3} u_i(*\mathbf{e_i})$, we have:

$$\begin{aligned}
d_2\mathbf{u} &= \sum_{i=1}^{3}(d_0u_i) \wedge (*\mathbf{e_i}) = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial u_i}{\partial x_j}\mathbf{e_i} \wedge (*\mathbf{e_i}) \\
&= \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i}\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3} = \sum_{i=1}^{3}(\nabla \cdot \mathbf{u})\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}
\end{aligned}$$

Now we will give a brief summary about differential k-form in $\mathbb{R}^3$ and the corresponding exterior derivatives in table 19.1.

Table 19.1. Differential k-form in $\mathbb{R}^3$ and exterior derivative d_k

k-form		proxy of d_k
0	$u(x)$	*grad*
1	$u_1(x)\mathbf{e_1} + u_2(x)\mathbf{e_2} + u_3(x)\mathbf{e_3}$	*curl*
2	$u_1(x)\mathbf{e_2} \wedge \mathbf{e_3} + u_2(x)\mathbf{e_3} \wedge \mathbf{e_1} + u_3(x)\mathbf{e_1} \wedge \mathbf{e_2}$	*div*
3	$u(x)\mathbf{e_1} \wedge \mathbf{e_2} \wedge \mathbf{e_3}$	

An important property of exterior derivative is $d_{i+1}d_i = 0$. It means that in $\mathbb{R}^3$:

$$\nabla \times (\nabla u) = 0,\ \nabla \cdot (\nabla \times \mathbf{u}) = 0$$

This property is the so-called Poincare theorem.

19.2.3 Stokes theorem

Another important thing about differential forms is Stokes theorem. After learning k-forms and exterior derivatives, we can move to Stokes theorem. A unified representation of Stokes theorem is

$$\int_{M_k} d_{k-1}\omega = \int_{\partial M_k} \omega$$

where M_k is an arbitrary k-manifold, ∂ is boundary operator, ω is a $(k-1)$-form. Next we will take $\mathbb{R}^3$ as example to see the concrete expressions of Stokes theorem.

- $k = 0$: $\int_c \nabla u = u(p) - u(q)$, where c is a curve in $\mathbb{R}^3$.
- $k = 1$: $\iint_S (\nabla \times \mathbf{u}) \cdot d\mathbf{A} = \int_{\partial S} \mathbf{u} \cdot d\mathbf{l}$.
- $k = 2$: $\iiint_V \nabla \cdot \mathbf{u} dx = \iint_{\partial V} \mathbf{u} \cdot d\mathbf{A}$.

Stokes theorem is of great importance in mathematics. The Stokes theorem hints us to prove the unisolvence of finite element elements in a unified way. We will explain what does it mean by unisolvence and how to construct the lowest order finite element basis for those forms in $\mathbb{R}^3$.

19.3 Finite elements in terms of differential forms

In this section, we will discuss finite element method for differential forms. We focus on $3D$ cases. We will give formal definitions related with FEM, discuss how to construct the lowest order elements for k-form in $3D$ and prove the unisolvence property of those finite element spaces.

Formal definitions

Before discussing about finite element method, we define Sobolev space for differential k-form first. The Hilbert space for k-form is defined as:

$$H(d_k; \Omega) = \{v \in L^2(\Omega) : \; d_k v \in L^2(\Omega)\}$$

where d_k is the exterior derivative. In $\mathbb{R}^3$, the Sobolev spaces for $k = 0, 1, 2, 3$ are the widely used Hilbert spaces $H(grad)$, $H(curl)$, $H(div)$ and L^2.

Next, we will give a formal definition of finite element method. A finite element method is defined by a triple $(K, \Sigma_K, \mathcal{P}_K)$:

- K is a simplex in $\mathbb{R}^d$, where $d = 2$ or 3.
- Σ_K is a set of degrees of freedom (or saying that Σ_K is a set of linear functionals).
- $\mathcal{P}_K$ is a space of polynomials on K s.t. Σ_K gives its dual basis.

Example This is an $1D$ example. Assume that:

- $K = [a, a + h]$ for a given $a \in \mathbb{R}$.
- $\Sigma_K = \{u(a), u(a + h)\}$.
- $\mathcal{P}_K = \{\text{linear polynomials}\}$.

then we know that $v \in \mathcal{P}_K$ looks like $v = \alpha + \beta x$.

An important property related with finite element space is unisolvence.

Definition 15 (Unisolvence). *Given a mesh element $K \subseteq \mathbb{R}^d$ ($d = 1, 2, 3$), $\Sigma_K = \{l_1, l_2, \cdots, l_m\}$ (a set of linear functionals), and $\mathcal{P}_K$ is a set of linear polynomials, we say that $(K, \Sigma_K, \mathcal{P}_K)$ is unisolvent if specifying a value for each degree of freedoms uniquely determines an element in $\mathcal{P}_K$.*

Given Σ_K and $\mathcal{P}_K$, to verify unisolvence, we only need to show that the implication holds: if $v \in \mathcal{P}_K$ satisfies

$$l_1(v) = l_2(v) = \cdots = l_m(v) = 0$$

then $v = 0$.

Example. Assume that:

- $K = [a, a + h]$ for a given $a \in \mathbb{R}$.
- $\Sigma_K = \{u(a), u(a + h)\}$.
- $\mathcal{P}_K = \{\text{linear polynomials}\}$.

If $v \in \mathcal{P}_K$, $v(a) = v(a+h) = 0$, then for any $x \in (a, a+h)$:

$$v(x) = v(\alpha a + \beta(a+h)) = \alpha v(a) + \beta v(a+h) = 0$$

so $v = 0$.

Another important property of $(K, \Sigma_K, \mathcal{P}_K)$ is that: if $(K, \Sigma_K, \mathcal{P}_K)$ is unisolvent, then there exists a basis $\{\phi_i\}$ in $\mathcal{P}_K$ s.t.

$$l_j(\phi_i) = \delta_{ij}$$

Now we know the formal definition of finite element space on an element K in a given mesh, which is usually called local finite elemnt space. Based on this, we can define global finite elemnt space.

How to construct basis

Next we will discuss how to construct the lowest order basis for Sobolev spaces $H(d_k; \Omega)$ in $\mathbb{R}^3$. According to previous discussion, we only need to construct a local finite element space, and glue them together. We use $(K, \Sigma_k, ||)$ to denote the local finite element space on element K for $H(d_k; \Omega)$. The criterions of our construction is as follows:

1. $\mathbb{R}$ or $\mathbb{R}^3$ is a subspace of $H(d_k; \Omega)$.
2. $d_k u$ is a constant number or constant vector.
3. The dimension of $\mathcal{P}_k$ is as small as possible.

Next, we will derive the basis functions for $k = 0, 1, 2$ in detail.

- $k = 0$. We consider $v_0 \in \mathcal{P}^1(K)$, i.e. $v_0 = a_0 + a_1x_1 + a_2x_2 + a_3x_3$. We know that

$$\nabla v = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

  is constant approximation of the gradient of the exact solution. So v_0 is the minimal polynomial we need to approximate $H(\text{grad}, \Omega)$. For this function we have 4 degrees of freedoms, corresponding to 4 vertices of a tetrahedron.
- $k = 1$. In this case, we want to have vector approximation $\underset{\sim}{v} \in \underset{\sim}{\mathcal{P}_1}(K)$. Here we consider

$$\underset{\sim}{v_1} = \underset{\sim}{a_0} + \underset{\sim}{u} \wedge \underset{\sim}{x} = \underset{\sim}{a_0} + \underset{\sim}{a} \times \underset{\sim}{x}.$$

  Since

$$\underset{\sim}{\nabla} \times \underset{\sim}{v_1} = 2\underset{\sim}{a},$$

  we know that this linear function has the minimal degree of freedoms. This linear function has 6 degrees of freedoms. It corresponds to the 6 edges of a tetrahedron.
- $k = 2$. For this case, we just need $\underset{\sim}{\nabla} \cdot \underset{\sim}{v}$ be scalar constant. Then we can take $\underset{\sim}{v_2} = \underset{\sim}{a_0} + a\underset{\sim}{x}$. Then

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v_2} = 3a.$$

  In this case, we have 4 degrees of freedoms. They correspond to 4 faces of a tetrahedron.

Unisolvence property

Next we will prove the unisolvence property of the lowest order finite element spaces constructed above:

$$\begin{aligned} \mathcal{P}_0(K) &= \{a_0 + a_1x_1 + a_2x_2 + a_3x_3 | a_i \in \mathbb{R}^1, 0 \le i \le 3\} \\ \mathcal{P}_1(K) &= \left\{\underset{\sim}{a_0} + \underset{\sim}{a_1} \times \underset{\sim}{x} | \underset{\sim}{a_0} \in \mathbb{R}^3, \underset{\sim}{a_1} \in \mathbb{R}^3\right\} \\ \mathcal{P}_2(K) &= \left\{\underset{\sim}{a_0} + a\underset{\sim}{x} | \underset{\sim}{a_0} \in \mathbb{R}^3, a \in \mathbb{R}^1\right\} \end{aligned} \tag{19.1}$$

- $k = 0$. $d_0 = \text{grad} = \underset{\sim}{\nabla}$. The dofs are

$$\mathcal{N}_0(K) = \{v(a_i), 0 \le i \le 3\}.$$

The unisolvence means that

$$v(a_i) = 0, \forall i \;\Rightarrow\; v \equiv 0,$$

or

$$\text{There exists a unique } v \in \mathcal{P}_0(K), \text{ s.t. } v(a_i) = y_i,\ 0 \le i \le 3.$$

Proof. Take any edge $[a_i, a_j]$. We do the following integral

$$\int_{[a_i,a_j]} \underset{\sim}{\nabla} v ds = v(a_j) - v(a_i) = 0.$$

Note that $\underset{\sim}{\nabla} v$ is constant vector. We know that $\underset{\sim}{\nabla} v \cdot (a_i - a_j) = 0, \forall i, j$.
Since $\underset{\sim}{\nabla} v$ only has three degrees of freedoms and it is perpendicular to 6 edges, 3 of which are linearly independent. So $\underset{\sim}{\nabla} v = 0$ and furthermore, $v = 0$. □

- $k = 1$. $d_1 = \text{curl}$. The dofs are

$$\mathcal{N}_1(K) = \left\{ \int_{[a_i,a_j]} \underset{\sim}{v} \cdot \underset{\sim}{\tau},\ i \ne j \right\}.$$

To prove the unisolvence, we need to show that

$$\int_{[a_i,a_j]} \underset{\sim}{v} \cdot \tau = 0, \forall i, j \;\Rightarrow\; \underset{\sim}{v} = 0.$$

Proof. Take any face of the tetrahedron $[a_i, a_j, a_k]$. By Stokes Theorem,

$$\iint_F \underset{\sim}{\nabla} \times \underset{\sim}{v} dN = \int_{\partial F} \underset{\sim}{v} \cdot \tau = 0.$$

Since $\underset{\sim}{\nabla} \times \underset{\sim}{v}$ is constant vector, we have

$$(\underset{\sim}{\nabla} \times \underset{\sim}{v}) \cdot n_F = 0,\ \forall F.$$

Here n_F is the normal vector of face F. We know that $\underset{\sim}{\nabla} \times \underset{\sim}{v}$ is perpendicular to the normal vectors of all the face, 3 of which are linearly independent. So $\underset{\sim}{v} = 0$. □

- $k = 2$. The dofs are

$$\mathcal{N}_2(K) = \left\{ \int_{[a_i,a_j,a_k]} \underset{\sim}{v} \cdot \underset{\sim}{n},\ \forall [a_i, a_j, a_k] \right\}.$$

To prove the unisolvence, we need to show that

$$\int_{[a_i,a_j,a_k]} \underset{\sim}{v} \cdot \underset{\sim}{n} = 0, \forall i, j, k \;\Rightarrow\; \underset{\sim}{v} = 0.$$

Proof. The proof is straightforward by Stokes Theorem

$$\iiint_K \underset{\sim}{\nabla} \cdot \underset{\sim}{v} = \iint_{\partial K} \underset{\sim}{v} \cdot \underset{\sim}{n_F} = 0.$$

Then we know that $\underset{\sim}{v} = 0$. □

In any case, the nodal parameters and the shape functions match perfectly by the Stokes theorem.

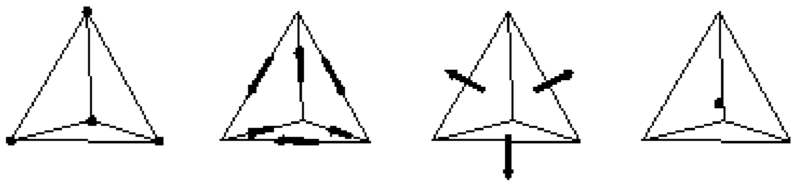

Fig. 19.2. Degrees of freedom for $H_h^1, H_h^{\text{curl}}, H_h^{\text{div}}$ and L_h^2 for $k = 0$.

19.4 Exact sequence and commuting diagram

The lowest order of the elements are defined as

$$
\begin{aligned}
H_h(\text{grad}, \Omega) &= \{v_h : v_h|_K \in P_1(K), [v_h]_F = 0, \forall K, \forall \text{face } F\} \subset H(\text{grad}, \Omega) = H^1(\Omega)\\
H_h(\text{curl}, \Omega) &= \{v_h : v_h|_K = \alpha + \beta \times \mathbf{x}, [v_h \times \mathbf{n}]_F = 0, \forall K, \forall \text{face } F \subset H(\text{curl}, \Omega)\\
H_h(\text{div}, \Omega) &= \{v_h : v_h|_K = \alpha + \beta\mathbf{x}, [v_h \cdot \mathbf{n}]_F = 0, \forall K, \forall \text{face } F\} \subset H(\text{div}, \Omega)\\
L_h^2(\Omega) = H_h(0, \Omega) &= \{v_h : v_h|_K = \alpha, \forall K\} \subset L^2(\Omega)
\end{aligned}
$$

The *degrees of freedom* can be defined as

$$
\begin{aligned}
\mathcal{N}_a(v) &= v(a_i) & 0 \le i \le 3\\
\mathcal{N}_e(v) &= \int_e v \cdot \tau & e = [a_i, a_j] 0 \le i < j \le 3\\
\mathcal{N}_F(v) &= \int_F v \cdot \mathbf{n} & F = [a_i, a_j, a_k] 0 \le i < j < k \le 3\\
\mathcal{N}_K(v) &= \int_K v & K = [a_0, a_1, a_2, a_3]
\end{aligned}
$$

Lemma 125. *For the sequence,*

$$
\mathbb{R}^1 \xrightarrow{1} H_h(\text{grad}) \xrightarrow{\text{grad}} H_h(\text{curl}) \xrightarrow{\text{curl}} H_h(\text{div }) \xrightarrow{\text{div}} L_h^2 \xrightarrow{1} \mathbb{R}^1
$$

This is an exact sequence, i.e. Range(left mapping)=Ker(right mapping).

Proof. First $\mathbb{R}^1 = \text{Ker}(\text{grad})$ is easy to prove.
Now we consider

$$
\text{grad} H_h(\text{grad}) = \text{Ker}(\text{curl}).
$$

Given

$$
\underset{\sim}{v} = \underset{\sim}{\alpha} + \underset{\sim}{\beta} \times \underset{\sim}{x} \in \text{Ker}(\text{curl}),
$$

we know that $\underset{\sim}{\beta} = 0$ due to $\text{curl}\underset{\sim}{v} = 0$.

Since $\underset{\sim}{v} \in H(\text{curl})$, we know that there exists $\phi \in H(\text{grad})$ such that

$$
\text{grad}\phi = \underset{\sim}{v} = \underset{\sim}{\alpha}.
$$

Note that it is constant vector element-wise. Then on each element (tetrahedron) we know that

$$\phi = \gamma + \underset{\sim}{\alpha} \cdot \underset{\sim}{x}.$$

Because we already have $\phi \in H(\text{grad})$, we know that $\phi \in H_h(\text{grad})$.

The next equation to prove is

$$\text{curl} H_h(\text{curl}) = \text{Ker}(\text{ div}).$$

Based on similar argument, we know that $\underset{\sim}{v} \in \text{Ker}(\text{div})$ will have the following form

$$\underset{\sim}{v} = \underset{\sim}{\alpha} + \beta \underset{\sim}{x}$$

and $\beta = 0$. Then $\phi \in H(\text{curl})$ satisfying $\text{curl}\phi = \underset{\sim}{v}$ will have the following form

$$\phi = \underset{\sim}{\gamma} + \frac{1}{2}\underset{\sim}{\alpha} \times \underset{\sim}{x}.$$

Therefore, $\phi \in H_h(\text{curl})$.

The last equation

$$\text{div } H_h(\text{div }) = \text{Ker}(1)$$

can be similarly proved.

$$\text{Ker}(1) = \{0\}.$$

We have $\phi \in H(\text{div })$ such that $\text{div } \phi = 0$. In this case $\phi = \underset{\sim}{\alpha} \in H_h(\text{div })$. □

Canonical interpolation

We define the following interpolations

$$\Pi_h^{\text{grad}} v = \sum_{a_i \in \text{nodes}} N_i(v)\phi_i(x)$$
$$\Pi_h^{\text{curl}} v = \sum_{e \in \text{edges}} N_e(v)\phi_e(x)$$
$$\Pi_h^{\text{div}} v = \sum_{F \in \text{faces}} N_F(v)\phi_F(x)$$

We need the "nodal" basis functions to satisfy the following properties

$$\begin{aligned} N_i(\phi_j) &= \delta_{i,j}, & \forall i, j &\in \text{nodes} \\ N_{e'}(\phi_e) &= \delta_{e,e'}, & \forall e, e' &\in \text{edges} \\ N_{F'}(\phi_F) &= \delta_{F',F}, & \forall F, F' &\in \text{faces}. \end{aligned}$$

In general, we define

$$\delta_{ab} = \begin{cases} 1, & \text{if } a = b \\ 0, & \text{if } a \neq b. \end{cases}$$

Lemma 126. *The following diagrams are all commutative.*

$$\begin{array}{ccccccccccc} R^1 & \xrightarrow{I} & C(\bar{\Omega}) \cap H(\text{grad}) & \xrightarrow{\textbf{grad}} & C(\bar{\Omega}) \cap H(\text{curl}) & \xrightarrow{\textbf{curl}} & C(\bar{\Omega}) \cap H(\text{div }) & \xrightarrow{div} & L^2(\Omega) & \longrightarrow & 0 \\ \downarrow I & & \downarrow \Pi_h^{grad} & & \downarrow \Pi_h^{curl} & & \downarrow \Pi_h^{div} & & \downarrow \Pi_h^0 & & \\ R^1 & \longrightarrow & H_h(\textbf{grad}) & \xrightarrow{\textbf{grad}} & H_h(\textbf{curl}) & \xrightarrow{\textbf{curl}} & H_h(div) & \xrightarrow{div} & L_h^2(\Omega) & \longrightarrow & 0. \end{array}$$

Proof. For any smooth function v, we need to prove that

$$\mathrm{grad}\Pi_h^{\mathrm{grad}} v = \Pi_h^{\mathrm{curl}} \mathrm{grad} v \in H_h(\mathrm{curl}).$$

In fact, on any edge $e = [a_i, a_j]$,

$$
\begin{aligned}
&= \int_e (\mathrm{grad}\Pi_h^{\mathrm{grad}} v)\tau_e \\
&= \int_e \frac{\partial \Pi_h^{\mathrm{grad}} v}{\partial \tau} = \Pi_h^{\mathrm{grad}} v|_{a_i}^{a_j} \\
&= \Pi_h^{\mathrm{grad}} v(a_j) - \Pi_h^{\mathrm{grad}}(a_i) \\
&= v(a_j) - v(a_i).
\end{aligned}
\tag{19.2}
$$

On the other side,

$$N_e(\Pi_h^{\mathrm{curl}}) = \int_e (\Pi_h^{\mathrm{curl}} \mathrm{grad} v)\tau_e = N_e(\mathrm{grad} v) = v(a_j) - v(a_i).$$

□

19.5 Applications in mixed finite elements

$$
\begin{array}{ccccccccccc}
R^1 & \xrightarrow{I} & C^1(\bar{\Omega}) & \xrightarrow{\mathbf{grad}} & C^1(\bar{\Omega}) & \xrightarrow{\mathbf{curl}} & C^1(\bar{\Omega}) & \xrightarrow{\mathrm{div}} & C^1(\bar{\Omega}) & \longrightarrow & 0 \\
\downarrow I & & \downarrow \Pi_h^{grad} & & \downarrow \Pi_h^{curl} & & \downarrow \Pi_h^{div} & & \downarrow \Pi_h^0 & & \\
R^1 & \longrightarrow & H_h(\mathbf{grad}, \Omega) & \xrightarrow{\mathbf{grad}} & H_h(\mathbf{curl}, \Omega) & \xrightarrow{\mathbf{curl}} & H_h(\mathrm{div}, \Omega) & \xrightarrow{\mathrm{div}} & L_h^2(\Omega) & \longrightarrow & 0.
\end{array}
$$

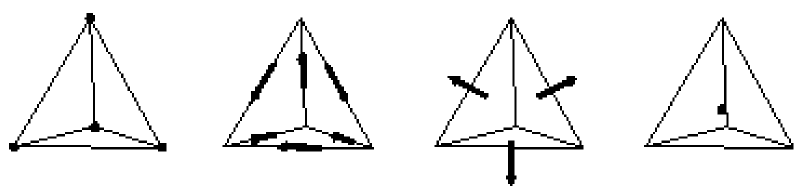

Fig. 19.3. Degrees of freedom for $H_h^1, H_h^{\mathrm{curl}}, H_h^{\mathrm{div}}$ and L_h^2 for $k = 0$.

The lowest order of the elements are defined as

$$
\begin{aligned}
H_h(\mathrm{grad}, \Omega) &= \{v_h : v_h|_K \in P_1(K), [v_h]_F = 0, \forall K, \forall \text{face } F\} \subset H(\mathrm{grad}, \Omega) = H^1(\Omega) \\
H_h(\mathrm{curl}, \Omega) &= \{v_h : v_h|_K = \alpha + \beta \times \mathbf{x}, [v_h \times \mathbf{n}]_F = 0, \forall K, \forall \text{face } F \subset H(\mathrm{curl}, \Omega) \\
H_h(\mathrm{div}, \Omega) &= \{v_h : v_h|_K = \alpha + \beta \mathbf{x}, [v_h \cdot \mathbf{n}]_F = 0, \forall K, \forall \text{face } F\} \subset H(\mathrm{div}, \Omega) \\
L_h^2(\Omega) = H_h(0, \Omega) &= \{v_h : v_h|_K = \alpha, \forall K\} \subset L^2(\Omega)
\end{aligned}
$$

The *degrees of freedom* can be defined as

$$
\begin{aligned}
\mathcal{N}_a(v) &= v(a_i) & 0 \le i \le 3\\
\mathcal{N}_e(v) &= \int_e v\cdot\tau & e=[a_i,a_j]0\le i<j\le 3\\
\mathcal{N}_F(v) &= \int_F v\cdot\mathbf{n} & F=[a_i,a_j,a_k]0\le i<j<k\le 3\\
\mathcal{N}_K(v) &= \int_K v & K=[a_0,a_1,a_2,a_3]
\end{aligned}
$$

Applications: Darcy's Law

By Darcy's Law, velocity is proportional to pressure gradient:

$$\underset{\sim}{u}=\alpha\nabla p.$$

The strong form is as follows.

$$
\begin{cases}
-\mathrm{div}\ (\alpha\nabla p)=f, & \\
p=0, & \text{on } \mathit{\Gamma}_D,\\
\dfrac{\partial p}{\partial n}=0, & \text{on } \mathit{\Gamma}_N.
\end{cases}
\tag{19.3}
$$

For this problem, there are three approaches.

- **Finite element for primal formulation**
  The variational form is: find $p\in\mathbb{V}_D:=\{p\in H^1(\mathit{\Omega}), p|_{\mathit{\Gamma}_D}=0\}$, s.t.

  $$(\alpha\nabla p,\nabla q)=(f,q),\quad \forall q\in\mathbb{V}_D.$$

  The corresponding discrete problem: find $p\in\mathbb{V}_D^h$, s.t.

  $$(\alpha\nabla p,\nabla q)=(f,q),\quad \forall q\in\mathbb{V}_D^h.$$

  This approach is not usually used in applications since it is not conservative.
- **Finite volumen method**

  $$\int_{\partial K'}\alpha\frac{\partial p_h}{h}=-\int_{K'}f,\quad \forall K'(\text{control volume}).$$

  Finite volume method is very much favored due to the conservation property.
- **Mixed finite element method**
  The variational formulation: find $p\in L^2(\mathit{\Omega}),\underset{\sim}{u}\in H_{\mathit{\Gamma}_N}(\mathrm{div})=\{\underset{\sim}{v}\in H(\mathrm{div}),\underset{\sim}{v}\cdot\mathbf{n}|_{\mathit{\Gamma}_N}=0\}$, such that

  $$
  \begin{cases}
  (\alpha^{-1}\underset{\sim}{u},\underset{\sim}{v})+(p,\mathrm{div}\underset{\sim}{v})=0, & \forall\underset{\sim}{v}\in H_{\mathit{\Gamma}_N}(\mathrm{div})\\
  (\mathrm{div}\underset{\sim}{u},q)=(f,q), & \forall q\in L^2(\mathit{\Omega})
  \end{cases}
  $$

  that is

  $$\begin{pmatrix}\alpha^{-1} & (\mathrm{div})^*\\ \mathrm{div} & 0\end{pmatrix}$$

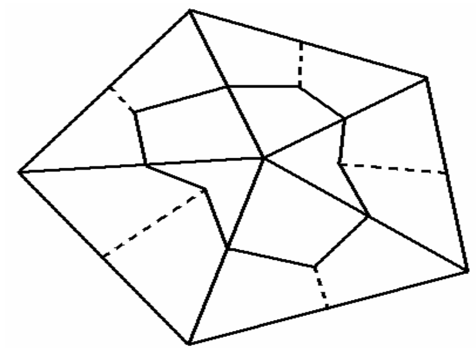

Fig. 19.4. Control volume

Remark 32. Note that if $\Gamma_D = \emptyset$, then we require $\int_\Omega f = 0$, since $\int_\Omega f = \int_\Omega -\mathrm{div}\, \underset{\sim}{v} = \int_{\partial\Omega} \underset{\sim}{v} \cdot n = 0$.

Mixed finite element approximation: find $\underset{\sim}{u}_h \in H_{\Gamma_N,h}(\mathrm{div})$, $p_h \in L^2_h(\Omega)$, such that

$$\begin{cases} (\alpha^{-1}\underset{\sim}{u}_h, \underset{\sim}{v}_h) + (p, \mathrm{div}\underset{\sim}{v}_h) = 0, & \forall \underset{\sim}{v}_h \in H_{\Gamma_N,h}(\mathrm{div}) \\ (\mathrm{div}\underset{\sim}{u}_h, q_h) = (f, q_h), & \forall q_h \in L^2_h(\Omega) \end{cases} \tag{19.4}$$

Mixed finite element is locally conservative: when $f = 0$.

$$(\mathrm{div}\underset{\sim}{u}_h, q_h) = 0 \Rightarrow \int_K \mathrm{div}\underset{\sim}{u}_h = 0 \Rightarrow \mathrm{div}\underset{\sim}{u}_h = 0.$$

As a matter of fact, $\int_{\partial G} \underset{\sim}{u}_h \cdot \mathbf{n} = 0, \forall G \subset \Omega$.

Well-posedness

Continuous problem:

$$\begin{cases} a(u,v) + b(v,p) & = \langle f, v\rangle, \quad \forall v \in \mathbb{V} \\ b(u,q) & = \langle g, q\rangle, \quad \forall q \in \mathbb{Q} \end{cases} \tag{19.5}$$

Define

$$\mathbb{V}_0 = \{v \in \mathbb{V} | b(v,q) = 0, \forall q\} = \{v \in \mathbb{V}, \mathrm{div}\ v = 0\}.$$

The coercivity is satisfied

$$a(\underset{\sim}{v}, \underset{\sim}{v}) = (\alpha^{-1}\underset{\sim}{v}, \underset{\sim}{v}) \geq \|\underset{\sim}{v}\|_0^2 + \|\mathrm{div}\ v\|_0^2.$$

Take $f = 0$, $(\mathrm{div}\underset{\sim}{u}_h, q_h) = 0$.

$$\mathrm{div} : H_{0,h}(\mathrm{div}) \overset{\text{onto}}{\longrightarrow} L^2_{h,0} =: \{v \in L^2, \int_\Omega v dx = 0\}.$$

If taking $q_h = \mathrm{div}\mathbf{u_h} \Rightarrow \mathrm{div}\underset{\sim}{u}_h = 0$. Take $\underset{\sim}{v}_h = \underset{\sim}{u}_h$. Then we have $(\mathrm{div}\ \underset{\sim}{v}_h, q_h) = 0, \forall \underset{\sim}{v}_h$.
But $\mathrm{div} : H_{0,h}(\mathrm{div}) \Rightarrow L^2_{h,0}$ is onto, choosing $\underset{\sim}{v}_h$ such that $\mathrm{div}\underset{\sim}{v}_h = p_h$, then $(p_h, p_h) = 0 \Rightarrow p_h = 0$.

Remark 33. Given $q \in L^2(\Omega)$, $\exists \underset{\sim}{v} \in H_{\Gamma_D}(\text{div})$, such that

$$\text{div}\underset{\sim}{v} = f, \text{ and } \|\underset{\sim}{v}\|_{H(\text{div})} \le c\|q\|_{L^2}.$$

Proof. Consider

$$\begin{cases} -\triangle\phi = q \\ \phi = 0, & \text{on } \Gamma_D \\ \dfrac{\partial\phi}{\partial\mathbf{n}} = 0, & \text{on } \Gamma_N. \end{cases} \tag{19.6}$$

Let $\underset{\sim}{v} = -\nabla\phi$, then $-\text{div}\underset{\sim}{v} = q$. By elliptic regularity,

$$\|\underset{\sim}{v}\|_{H(\text{div})} \le \|\phi\|_2 \le c\|q\|_0.$$

□

Remark 34. Consequently we have that $\forall q \in L^2(\Omega)$

$$\frac{(\text{div } \underset{\sim}{v}, q)}{\|v\|_{H(div)}} \ge c\|q\|_{L^2}.$$

Then by Brezzi theorem, the continuous mixed formulation is well-posed.

Discrete problem:

- $a(v_h, v_h) \ge \|v_h\|^2_{H(\text{div})}$, $\forall v_h \in V_{h,0} = \{v_h \in V_h, \text{div}v_h = 0\}$.
- Inf-sup condition. To prove it, we need $\Pi_h^0\text{div } \underset{\sim}{v} = \text{div } \Pi_h^{div}\underset{\sim}{v}$.
  We know that $\forall q_h \in \mathbb{Q}_h$, we can find $\underset{\sim}{v} \in \mathbb{V}$ s.t.

$$\text{div } \underset{\sim}{v} = q_h, \quad \frac{(\text{div } \underset{\sim}{v}, q_h)}{\|v\|_{H(div)}} \ge c\|q_h\|_{L^2}.$$

We consider $\underset{\sim}{v_h} = \Pi_h^{div}\underset{\sim}{v}$. Then,

$$(\text{div } \underset{\sim}{v_h}, q_h) = (\Pi_h^0\text{div } \underset{\sim}{v}, q_h) = (\Pi_h^0 q_h, q_h) = \|q_h\|^2$$

$$\|\underset{\sim}{v_h}\|_{H(div)} = \|\Pi_h^{div}\underset{\sim}{v}\|_{H(div)}$$

Since $\Pi_h^{div}\underset{\sim}{v} = \sum_{l=1}^{NF} \int_{f_l} \underset{\sim}{n}_{f_l} \cdot \underset{\sim}{v} ds \phi_l$, where $\underset{\sim}{n}_{f_l}$ is the unit normal vector of face f_l, NF is the total number of faces, and ϕ_l is the lth lowest order basis of $H_h(div)$. In terms of barycentric coordination λ_i $(i = 0, 1, 2, 3)$, we know that for face $f_l = [a_i, a_j, a_k]$ $(0 \le i < j < k \le 3)$:

$$\phi_l = 2(\lambda_i\nabla\lambda_j \times \nabla\lambda_k + \lambda_j\nabla\lambda_k \times \nabla\lambda_i + \lambda_k\nabla\lambda_i \times \nabla\lambda_j)$$

And

$$div\phi_l = 2[\nabla\lambda_i \cdot (\nabla\lambda_j \times \nabla\lambda_k) + \nabla\lambda_j \cdot (\nabla\lambda_k \times \nabla\lambda_i) + \nabla\lambda_k \cdot (\nabla\lambda_i \times \nabla\lambda_j)]$$

We also know that:

$$\max_{x\in T} \lambda_i \le 1,\ \|\nabla\lambda_i\| \le \frac{1}{h_{min}},\ i = 0,1,2,3$$

where T is an arbitrary tetrahedron in the mesh, h_{min} is the smallest edge size in the mesh. So we can conclude that $\|\phi_i\|_0 \le Ch^{-1/2}$.
First we consider the L^2 norm

$$\begin{aligned}
\|\Pi_h^{\rm div}\, v\|_0^2 &= \left\|\sum_i N_i^{\rm div}\,(v)\phi_i\right\|_0^2 \\
&\le \sum_T \sum_{e\in\partial T} \left|\int_e v\cdot n\right|^2 \|\phi_i\|_0^2 \\
&\le \sum_T \sum_{e\in\partial T} h^2 \int_e |v|^2 h^{-1} \\
&\le \sum_T \|v\|_{1,T}^2 \\
&\le \|v\|_{1,\Omega}^2.
\end{aligned} \tag{19.7}$$

And we know that

$$\|{\rm div}\,\Pi_h^{\rm div}\, v\|_0^2 = \|P_0{\rm div}\, v\|_0^2 \le \|{\rm div}\, v\|_0^2 \le \|\nabla v\|_0^2.$$

We can conclude that

$$\|\underset{\sim}{v_h}\|_{H(div)} \le \|\underset{\sim}{v}\|_{H^1(\Omega)} \le \|\phi\|_{H^2(\Omega)} \le c\|q_h\|_{L^2(\Omega)}$$

These will finish our proof. Note that ϕ is the solution to Equation (19.6).

19.6 Integral form of Maxwell equation and boundary condition

In this section, we will give a brief introduction to integral form of Maxwell equation and its correpsonding boundary condition. Maxwell equation comes from four basic laws in physics: Gauss's law (for electric field and magnetic field), Faraday's law and Ampere's law.

19.6.1 Integral form of Maxwell's equations

Gauss's law for electric field says that the total of the electric flux out of a closed surface is equal to the total charge enclosed. In integral form, it can be written as:

$$\oint_{\partial V} \boldsymbol{D}\cdot {\rm d}\boldsymbol{S} = \int_V \rho dV$$

where V is an arbitrary volume in the space, ∂V is a closed surface, d$\mathbf{S}$ is the elementary area on the closed surface, $\boldsymbol{D}$ is electric displacement, ρ is the charge density within V. For linear media, $\boldsymbol{D} = \varepsilon\boldsymbol{E}$. ε is the so-called electric permittivity, $\boldsymbol{E}$ is electric field.
Gauss's law for magnetism says that the magnetic flux through any closed surface is zero, i.e.

$$\oint_S \boldsymbol{B}\cdot {\rm d}\boldsymbol{S} = 0$$

where S is a closed surface, $\boldsymbol{B}$ is the density of magnetic field (which is usually called magnetic field).
Faraday's law says that:

$$\oint_{\partial S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = -\int_S \frac{\partial \boldsymbol{B}}{\partial t} \cdot \mathrm{d}\boldsymbol{S}$$

where S is an arbitrary surface, ∂S is the boundary of S, $\mathrm{d}\boldsymbol{l}$ is the elementary length on the contour ∂S.

Ampere's law says that

$$\oint_{\partial S} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \cdot d\boldsymbol{S}$$

where $\boldsymbol{H}$ is the magnetic field (which is usually called magnetism), $\boldsymbol{J}$ is the current, and $\boldsymbol{D}$ is the electric displacement. And $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$, where μ is magnetic permeability. Besides $\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$, another constitutive relation is Ohms law:

$$\boldsymbol{J} = \sigma\boldsymbol{E} + \boldsymbol{J}_a$$

where σ is called the conductivity, which is a non-negative function of position. The vector function $\boldsymbol{J}_a$ describes the applied current density.

19.6.2 Interface or boundary condition of Maxwell's equations

The above four equations form Maxwell's equations, which are the integral form. Corresponding to those four equations, we get four boundary conditions. We will discuss those cases in the following subsections.

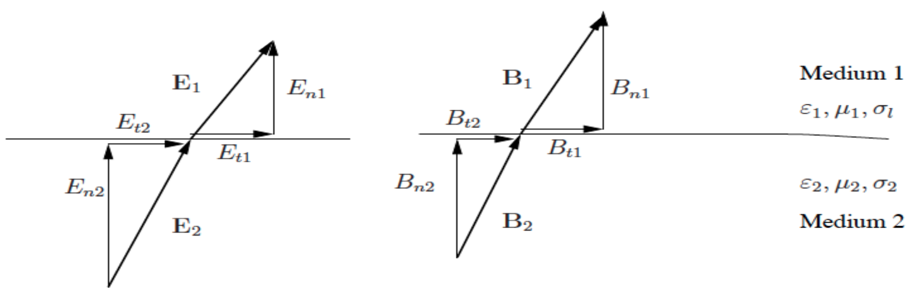

Fig. 19.5. Normal and tangential components illustrated for the cases of the $\boldsymbol{E}$ field and the $\boldsymbol{B}$ field.

Normal component of $\boldsymbol{D}$

By Gauss's law of electricity, we can get the boundary condition for the normal component of the electric displacement. Applying Gauss's law:

$$\oint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} = \int_V \rho dV$$

to a small 'pill-box', positioned such that the boundary sits between its 'upper' and 'lower' surfaces as shown in the illustration. If we shrink the side wall Δh to zero (keeping the interface sandwiched between the upper and lower surfaces), then all electric flux enters or leaves the pill-box through the top and bottom surfaces, and:

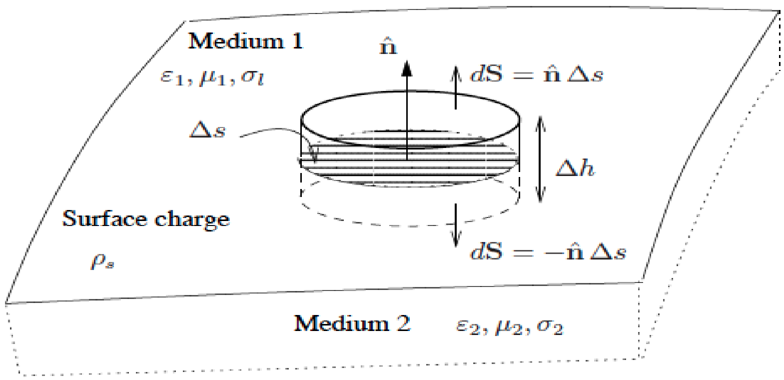

Fig. 19.6. Gauss pill box straddling the interfaces between two media

$$\oint_{\partial V} \boldsymbol{D} \cdot \mathrm{d}\boldsymbol{S} \to \boldsymbol{D}_1 \cdot \hat{\boldsymbol{n}}\Delta s + \boldsymbol{D}_2 \cdot (-\hat{\boldsymbol{n}})\Delta s = D_{n1}\Delta s - D_{n2}\Delta s$$

where D_{n1} and D_{n2} are the normal components of the flux density vector immediately on either side of the boundary in medium 1 and 2, and Δs is the elemental surface area.

The amount of charge enclosed as $\Delta h \to 0$ depends on whether there exists a layer of charge on the surface (i.e. an infinitesimally thin layer of charge). If a surface charge layer exists then:

$$\int_V \rho dV = \rho_s \Delta s$$

Therefore,

$$D_{n1}\Delta s - D_{n2}\Delta s = \rho_s \Delta s$$

which is equivalent to say $D_{n1} - D_{n2} = \rho_s$. If there is no surface charge, i.e. $\rho_s = 0$, we get $D_{n1} = D_{n2}$. In terms of electric field, it is written as $\varepsilon_1 E_{n1} = \varepsilon_2 E_{n2}$.

Normal component of B

The boundary condition for the normal component of the magnetic field can be obtained by applying Gauss's law:

$$\oint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} = 0$$

to a small pill-box. If we shrink the side wall Δh to zero, all magnetic flux and leaves (or enters) the pill-box through the top two surfaces,

$$\oint_{\partial V} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{S} \to \boldsymbol{B}_1 \cdot \hat{\boldsymbol{n}}\Delta s + \boldsymbol{B}_2 \cdot (-\hat{\boldsymbol{n}})\Delta s = B_{n1}\Delta s - B_{n2}\Delta s$$

equals zero. So we get:

$$B_{n1} - B_{n2} = 0$$

It means that the normal component of $\boldsymbol{B}$ is continuous at the boundaries.

Tangential component of $\boldsymbol{E}$

We can derive the tangential component of $\boldsymbol{E}$ by applying Faraday's law to a small rectangular loop positioned across the boundary, and in the plane of E_1 and E_2, as illustrated in the diagram 19.7 below. Consider the limiting case where the sides Δh perpendicular to boundary are allowed shrink to zero. In the

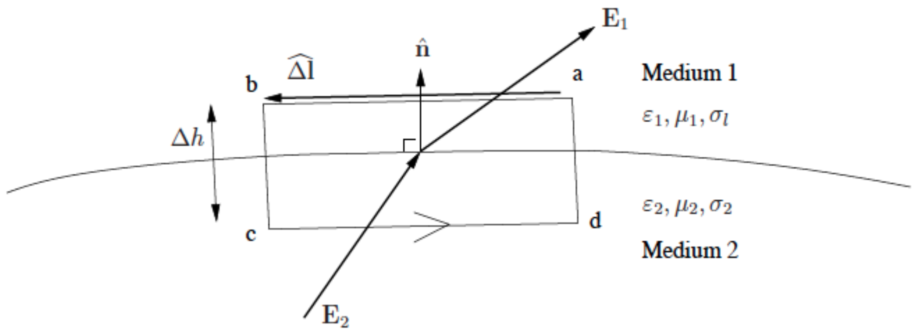

Fig. 19.7. To determine the boundary condition on the tangential component of the $\boldsymbol{E}$ field, Faraday's law is applied to rectangular loop straddling the interface between two media.

limit as $h \to 0$, the magnetic flux threading the loop shrinks to zero, and thus

$$\oint_{\partial S} \boldsymbol{E} \cdot d\boldsymbol{l} \to \int_a^b \boldsymbol{E}_1 \cdot d\boldsymbol{l} + \int_c^d \boldsymbol{E}_2 \cdot d\boldsymbol{l} = 0$$

It means that

$$\boldsymbol{E}_1 \cdot \Delta\boldsymbol{l} + \boldsymbol{E}_2 \cdot (-\Delta\boldsymbol{l}) = 0$$

Writing the tangential components of $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ along the contour as E_{t1} and E_{t2}, we have:

$$E_{t1} = E_{t2}$$

i.e. tangential components immediately on either side of a boundary are equal.

Tangential component of $\boldsymbol{H}$

We can derive the tangential component of $\boldsymbol{H}$ by applying Ampere's law to a closed loop as illustrated in 19.8. Again, the rectangular loop is in the plane of vectors $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$. Ampere's law says that

$$\oint_{\partial S} \boldsymbol{H} \cdot \mathrm{d}\boldsymbol{l} = \int_S (\boldsymbol{J} + \frac{\partial \boldsymbol{D}}{\partial t}) \cdot d\boldsymbol{S}$$

Consider the limiting case where the sides Δh perpendicular to the boundary are allowed shrink to zero. The left hand side becomes

$$\oint_{\partial S} \boldsymbol{H} \cdot d\boldsymbol{l} \to \int_a^b \boldsymbol{H}_1 \cdot d\boldsymbol{l} + \int_c^d \boldsymbol{H}_2 \cdot d\boldsymbol{l}$$

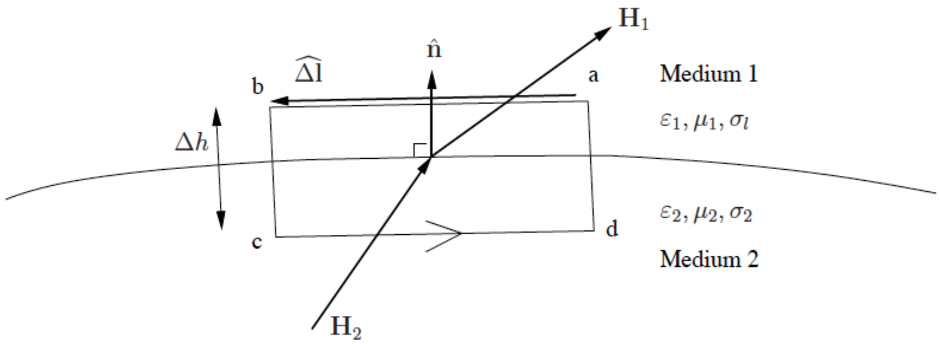

Fig. 19.8. To determine the boundary condition on the tangential component of the $\boldsymbol{H}$ field, Ampere's law is applied to rectangular loop straddling the interface between two media.

On the right hand side, the displacement current term $I_d = \int_S \frac{\partial \boldsymbol{D}}{\partial t} \cdot d\boldsymbol{S}$ shrinks to zero. For physical media, the conductivity σ is finite, and $\boldsymbol{J}_f$ is also finite. Thus within the loop $I_c = \int_S \boldsymbol{J} \cdot d\boldsymbol{S}$ also shrinks to zero, and so

$$\boldsymbol{H}_1 \cdot \Delta \boldsymbol{l} + \boldsymbol{H}_2 \cdot (-\Delta \boldsymbol{l}) = 0$$

which implies the tangential component of $\boldsymbol{H}$ does not change immediately on either side of the boundary, i.e.

$$H_{t1} = H_{t2}$$

But sometimes there may exist surface current, in this case we use J_s to represent the magnitude of the surface current density. And the boundary condition on the tangential component of $\boldsymbol{H}$ will become:

$$H_{t1} - H_{t2} = J_s$$

Next, we will give some examples to show the boundary condition for Maxwell's equations in practical problems.

Example 10. Let us consider the case of two media with different electric and magnetic properties seperated by a surface S with unit normal $\boldsymbol{\nu}$ pointing from region 2 to region 1 (figure 19.9). If the side of surface S

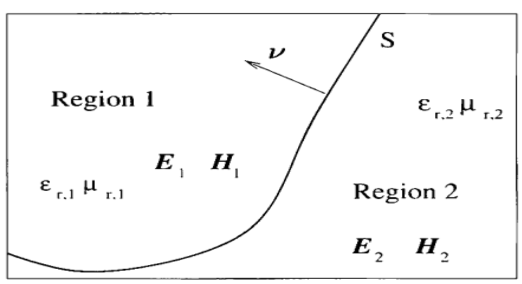

Fig. 19.9. Geometry of interface and subdomains.

labeled 2 in figure 19.9 is perfect conductor, then $\boldsymbol{E}_2 = 0$. So we can get the perfect conducting boundary condition for $\boldsymbol{E}_1$ and $\boldsymbol{B}_1$, i.e.

$$\boldsymbol{\nu} \times \boldsymbol{E}_1 = 0, \ \textit{on } S$$
$$\boldsymbol{\nu} \cdot \boldsymbol{B}_1 = 0, \ \textit{on } S$$

If the side of surface S labeled 2 is not perfect conductor, but allows the field to penetrate only a small distance, a more appropriate boundary condition is the impedence or imperfectly conducting boundary condition. The boundary condition in this case should be:

$$\boldsymbol{\nu} \times \boldsymbol{H}_1 - \lambda(\boldsymbol{\nu} \times \boldsymbol{E}) \times \boldsymbol{\nu} = 0$$

where λ is the impedence, which is a positive function of position of the surface of the material.

Example 11 (Radiation conditions). When the outer boundary of a domain recedes to infinity the domain is called unbounded or open. A condition must be specified at this outer boundary to obtain a unique solution for the problem. Such a condition is referred to as a radiation condition.

Assuming that all sources and objuects are immersed in free space and located within a finite distance from the origin of a coordinate system, the electric and magnetic fields are required to satisfy:

$$\lim_{r\to\infty} r\left[\nabla \times \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix} + ik_0 \boldsymbol{r} \times \begin{pmatrix} \boldsymbol{E} \\ \boldsymbol{H} \end{pmatrix}\right] = 0$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $\boldsymbol{r}$ is a unit normal vector in direction of radiance. This condition is usually referred to as the Sommerfeld radiation condition for general three-dimensional fields. This condition is exactly valid at infinity. In nuerical analysis, it is often desirable to reduce the size of a computational domain by using a finite boundary to truncate the infinite domain. When applied at such a finite boundary, the condition can be regarded as the lowest order radiation condition with limited accuracy.

19.7 Time domain Maxwell's equations

Based on the integral form of Maxwell's equations and Stoke theorem, we can get the following differential forms of Maxwell's equations:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0$$
$$\nabla \cdot \boldsymbol{D} = \rho$$
$$\frac{\partial \boldsymbol{D}}{\partial t} - \nabla \times \boldsymbol{H} = -\boldsymbol{J}$$
$$\nabla \cdot \boldsymbol{B} = 0$$

This is the so-call time domain Maxwell's equations. Since $\boldsymbol{D} = \varepsilon \boldsymbol{E}$, $\boldsymbol{H} = \mu^{-1}\boldsymbol{B}$, the time domain Maxwell's equations can be written as:

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \times \boldsymbol{E} = 0 \tag{19.8}$$
$$\nabla \cdot \varepsilon \boldsymbol{E} = \rho \tag{19.9}$$
$$\frac{\partial \varepsilon \boldsymbol{E}}{\partial t} - \nabla \times \mu^{-1}\boldsymbol{B} = -\boldsymbol{J} \tag{19.10}$$
$$\nabla \cdot \boldsymbol{B} = 0 \tag{19.11}$$

19.8 Frequency domain Maxwell's equations

Assume that electromagnetic propagation is plane wave of single frequency, then the variables $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$, $\boldsymbol{B}$ can be written as:

$$
\begin{aligned}
\boldsymbol{E}(\boldsymbol{x},t) &= \Re(\exp(i\omega t)\boldsymbol{E}(\boldsymbol{x}))\\
\boldsymbol{D}(\boldsymbol{x},t) &= \Re(\exp(i\omega t)\boldsymbol{D}(\boldsymbol{x}))\\
\boldsymbol{H}(\boldsymbol{x},t) &= \Re(\exp(i\omega t)\boldsymbol{H}(\boldsymbol{x}))\\
\boldsymbol{B}(\boldsymbol{x},t) &= \Re(\exp(i\omega t)\boldsymbol{B}(\boldsymbol{x}))\\
\boldsymbol{J}(\boldsymbol{x},t) &= \Re(\exp(i\omega t)\boldsymbol{J}(\boldsymbol{x}))\\
\rho(\boldsymbol{x},t) &= \Re(\exp(i\omega t)\rho(\boldsymbol{x}))
\end{aligned}
$$

So the time domain Maxwell's equations become:

$$
\begin{aligned}
i\omega\boldsymbol{B} + \nabla\times\boldsymbol{E} &= 0\\
\nabla\cdot\boldsymbol{D} &= \rho\\
i\omega\boldsymbol{D} - \nabla\times\boldsymbol{H} &= -\boldsymbol{J}\\
\nabla\cdot\boldsymbol{B} &= 0
\end{aligned}
$$

Since $\boldsymbol{D} = \varepsilon\boldsymbol{E}$, $\mu^{-1}\boldsymbol{B} = \boldsymbol{H}$ (ε is called permittivity, μ is called permeablity. Both vary for different materials). The equations becomes:

$$
\begin{aligned}
i\omega\boldsymbol{B} + \nabla\times\boldsymbol{E} &= 0 \quad (19.12)\\
\nabla\cdot\varepsilon\boldsymbol{E} &= \rho \quad (19.13)\\
i\omega\varepsilon\boldsymbol{E} - \nabla\times\mu^{-1}\boldsymbol{B} &= -\boldsymbol{J} \quad (19.14)\\
\nabla\cdot\boldsymbol{B} &= 0 \quad (19.15)
\end{aligned}
$$

19.9 Rational discretization

Now we understand Maxwell's equations from the perspective of physics: we can derive Maxwell's equations and impose appropriate boundary condition for different physical model. In this section, we will discuss how to discretize Maxwell's equations. There are many different ways to discretize it. Here, we focus on proposing a rational discretization for Maxwell's equations.

19.9.1 Time domain Maxwell's equations

In order to discrete Maxwell's equations (19.8)-(19.9), people usually pick one variable they want to solve and eliminate the rest. For example, for equation (19.8) and (19.10), if choosing to solve for electric field $\boldsymbol{E}$ and eliminating variable $\boldsymbol{B}$, we can obtain a second order equation as follows:

$$\varepsilon\frac{\partial^2\boldsymbol{E}}{\partial t^2} + \nabla^{-1}\nabla\times\boldsymbol{E} = -\frac{\partial\boldsymbol{J}}{\partial t}$$

If choosing to solve for magnetic field $\boldsymbol{B}$ and eliminating variable $\boldsymbol{E}$, we can get a second order equation:

$$\frac{\partial^2\boldsymbol{B}}{\partial t^2} + \nabla\times\varepsilon^{-1}\nabla\times\mu^{-1}\boldsymbol{B} = -\nabla\times\varepsilon\boldsymbol{J}$$

Noticing that $\boldsymbol{D}$ and $\boldsymbol{B}$ are 2-form, $\boldsymbol{E}$ and $\boldsymbol{H}$ are 1-form, it means that $\boldsymbol{E}$, $\boldsymbol{H}$ should be in $H(curl)$, while $\boldsymbol{B}$, $\boldsymbol{D}$ in $H(div)$. So we can conclude that the 2nd order equation for $\boldsymbol{E}$ is fine, but that for $\boldsymbol{B}$ is problematic. In order to make sure that $\boldsymbol{E} \in H(curl)$, $\boldsymbol{B} \in H(div)$, we have to work on the original system. A semi-discretization is as follows:

$$\frac{\boldsymbol{B}^{n+1} - \boldsymbol{B}^n}{\Delta t} + \nabla \times \boldsymbol{E}^{n+1} = 0$$
$$\frac{\varepsilon \boldsymbol{E}^{n+1} - \varepsilon \boldsymbol{E}^n}{\Delta t} - \nabla \times \mu^{-1} \boldsymbol{B}^{n+1} = -\boldsymbol{J}$$

So the full discretization should be: find $(\boldsymbol{E}_h^{n+1}, \boldsymbol{B}_h^{n+1}) \in H_h(curl) \times H_h(div)$ s.t.

$$(\frac{\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^n}{\Delta t}, \boldsymbol{v}_h) + (\nabla \times \boldsymbol{E}_h^{n+1}, \boldsymbol{v}_h) = 0, \ \forall \boldsymbol{v}_h \in H_h(div)$$
$$(\frac{\varepsilon \boldsymbol{E}_h^{n+1} - \varepsilon \boldsymbol{E}_h^n}{\Delta t}, \boldsymbol{w}_h) - (\mu^{-1} \boldsymbol{B}_h^{n+1}, \nabla \times \boldsymbol{w}_h) = -(\boldsymbol{J}, \boldsymbol{w}_h), \ \forall \boldsymbol{w}_h \in H_h(curl)$$

The corresponding boundary condition is:

$$\boldsymbol{n} \times \boldsymbol{E} = 0, \ on\ \partial\Omega$$
$$\boldsymbol{n} \cdot \boldsymbol{B} = 0, \ on\ \partial\Omega$$

This is what we mean by rational discretization.

Theorem 122. *If* $\nabla \cdot \boldsymbol{B}_h^0 = 0$*, then* $\nabla \cdot \boldsymbol{B}_h^n = 0$*, for any n.*

Proof. Since

$$(\frac{\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^n}{\Delta t}, \boldsymbol{v}_h) + (\nabla \times \boldsymbol{E}_h^{n+1}, \boldsymbol{v}_h) = 0, \ \forall \boldsymbol{v}_h \in H_h(div)$$

we can pick

$$\boldsymbol{v}_h = \frac{\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^n}{\Delta t} + \nabla \times \boldsymbol{E}_h^{n+1}$$

So we can conclude that $\frac{\boldsymbol{B}_h^{n+1} - \boldsymbol{B}_h^n}{\Delta t} + \nabla \times \boldsymbol{E}_h^{n+1} = 0$ almost everywhere. By mathematical induction, we can conclude that $\nabla \cdot \boldsymbol{B}_h^n = 0$, for any n, if $\nabla \cdot \boldsymbol{B}_h^0 = 0$. □

19.9.2 Frequency domain Maxwell's equations

The discretization of frequency domain Maxwell's equations is similar to that of time domain problem. For equation (19.12) and (19.14), if eliminating variable $\boldsymbol{B}$, we can get the second order equation for electric field $\boldsymbol{E}$:

$$\nabla \times \mu^{-1} \nabla \times \boldsymbol{E} - \omega^2 \varepsilon \boldsymbol{E} = -i\omega \boldsymbol{J}$$

If eliminating variable $\boldsymbol{E}$, we can get second order equation for magnetic field $\boldsymbol{B}$:

$$\nabla \times \varepsilon^{-1} \nabla \times \mu^{-1} \boldsymbol{B} - \omega^2 \boldsymbol{B} = \nabla \times \varepsilon^{-1} \boldsymbol{J}$$

In order to achieve rational discretization for frequency domain Maxwell's equations, we work on the original equations and apply the same discretization technique with time domain case. The discretization is as follows: find $(\boldsymbol{E}_h, \boldsymbol{B}_h) \in H_h(curl) \times H_h(div)$ s.t.

$$i\omega(\boldsymbol{B}_h, \boldsymbol{v}_h) + (\nabla \times \boldsymbol{E}_h, \boldsymbol{v}_h) = 0, \ \forall \boldsymbol{v}_h \in H_h(div)$$
$$i\omega(\varepsilon \boldsymbol{E}_h, \boldsymbol{w}_h) - (\mu^{-1} \boldsymbol{B}_h, \nabla \times \boldsymbol{w}_h) = -(\boldsymbol{J}, \boldsymbol{w}_h), \ \forall \boldsymbol{w}_h \in H_h(curl)$$